A New Code Based Public Key Encryption and Signature Scheme based on List Decoding

Presented by
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joint work with:
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Acknowledgements

- To **Tanja Lange** for inviting me to attend the workshop *Post-Quantum Cryptography and Quantum Algorithms,* Lorentz Center**,** 5-9 Nov 2012, where I got the initial idea
- To **Christiane Peters** for discussing with me the initial idea and for her encouragements
- To **Ludovic Perret** and **Jean-Charles Faugère** for inviting me to be a guest at their department at LIP6 for 3 months to analyze the scheme with Groebner bases
- To **Nicolas Sendrier** (discussing cheap distinguishers for the encryption variant of the scheme) and **Jean-Pierre Tillich** (discussing the scheme and the statistical properties of the signature part of the scheme) for spending two days at INRIA at Paris-Rocquencourt in discussions with us about diferent aspects of the security of the scheme

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Introduction

- In 1978 Robert McEliece proposed a public key scheme based on Coding Theory
- The scheme could be used only for encryption
- Difficulty of decoding random linear codes
- Not so attractive as RSA (public key bigger than 32KB)
- Scheme was analyzed 35 years and seems solid
  - Some parameters adjusted in 2008

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McEliece Public-Key Scheme

- **Key Generation**
  - *Alice* chooses ***S*, *G*** and ***P***
    - ***S*** is a random ($k \times k$) nonsingular binary matrix.
    - ***G*** is a ($k \times n$) generator matrix of a t-error-correcting binary linear code.
    - ***P*** is a random ($n \times n$) permutation matrix.
  - *Alice*'s secret key : ***S*, *G*** and ***P***
  - *Alice*'s public key : ***G' = SGP***

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McEliece Public-Key Scheme (Cont.)

- **Encryption**
  - *Bob* sends a k-bit binary message $\boldsymbol{m}$ to *Alice,* by computing

    $\boldsymbol{c} = \boldsymbol{mG}' + \boldsymbol{e}$

    - $\boldsymbol{e}$ is an n-bit random error vector of weight t.

McEliece Public-Key Scheme (Cont.)

- **Decryption**
  - When *Alice* receives $\boldsymbol{c}$, she
    1. Calculates $\boldsymbol{c}' = \boldsymbol{cP}^{-1} = \boldsymbol{mSG} + \boldsymbol{eP}^{-1}$
    2. Uses the decoding algorithm of the original code $\boldsymbol{G}$ to obtain $\boldsymbol{m}' = \boldsymbol{mS}$ form $\boldsymbol{c}'$
    3. Recovers message $\boldsymbol{m}$ by computing $\boldsymbol{m} = \boldsymbol{m}'\boldsymbol{S}^{-1}$

McEliece Public-Key Scheme (Cont.)

Parameters

- For 80 bits of security McEliece originally suggested to use Goppa codes and the security parameter sizes of ***n=1024, k=524, t=50***
- Recent analysis (Bernstein, Lange and Peters, 2008) suggests parameter sizes of ***n=2048, k=1751, t=27*** for 80 bits of security (standard algebraic decoding for the Goppa codes)
- Or ***n=1632, k=1269, t=34*** when using **list decoding** for the Goppa code

McEliece Public-Key Scheme (Cont.)

Q: Why Bob must produce just so little errors when he encrypts the messages?

Parameters

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How the codes are decoded in Coding Theory?

The set of all codewords composes the code C

•

How the codes are decoded in Coding Theory?

d – Minimum distance of the code C

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How the codes are decoded in Coding Theory?

•

$t=d/2$

radius for unique decoding of the code C

How the codes are decoded in Coding Theory?

-

$t=d/2$

How the codes are decoded in Coding Theory?

When the dimension of the codewords is $\boldsymbol{n}$, and their number is $\mathbf{2}^{k}$, and the minimum distance is $\boldsymbol{d}$, we talk about $(\boldsymbol{n}, \boldsymbol{k}, \boldsymbol{d})$ codes.

-

$t=d/2$

How the codes are decoded in Coding Theory?

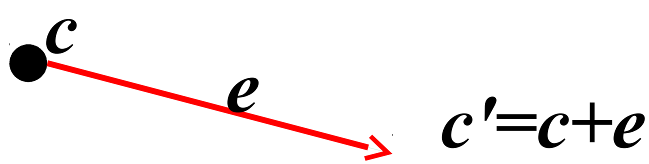
$t{=}d/2$ is constrained by:

$$k \leq n - \log_2\left(\sum_{i=0}^{t}\binom{n}{i}\right)$$

•

$t{=}d/2$

How the codes are decoded in Coding Theory?



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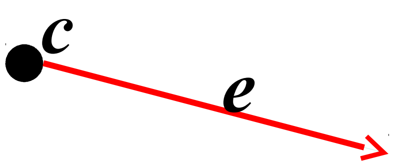

$$c' = c + e$$

c' can not be decoded if $||e|| > t$

Can we do better with some other decoding approach?

Can we do better with some other decoding approach?

- Yes we can :-)
  - By using **List decoding**
  - Proposed in 50’ by Elias and Wozencraft (but no efficient algorithm)
  - Sudan in 1997 proposed an efficient list decoding algorithm with polynomial run-time
  - Guruswami and Sudan in 1998 significantly improved Sudan’s algorithm

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Can we do better with some other decoding approach?

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Can we do EVEN BETTER than the classical list decoding, if our coding/decoding is for cryptographical purposes?

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Can we do EVEN BETTER than the classical list
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YES !

Can the
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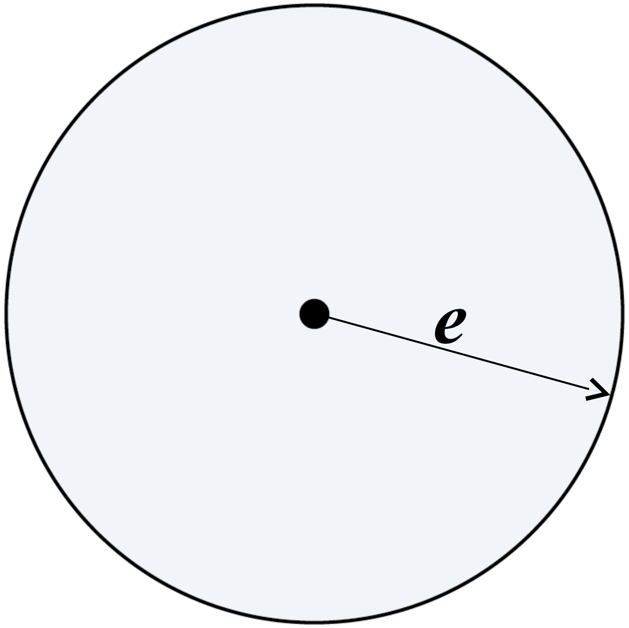
Our approach

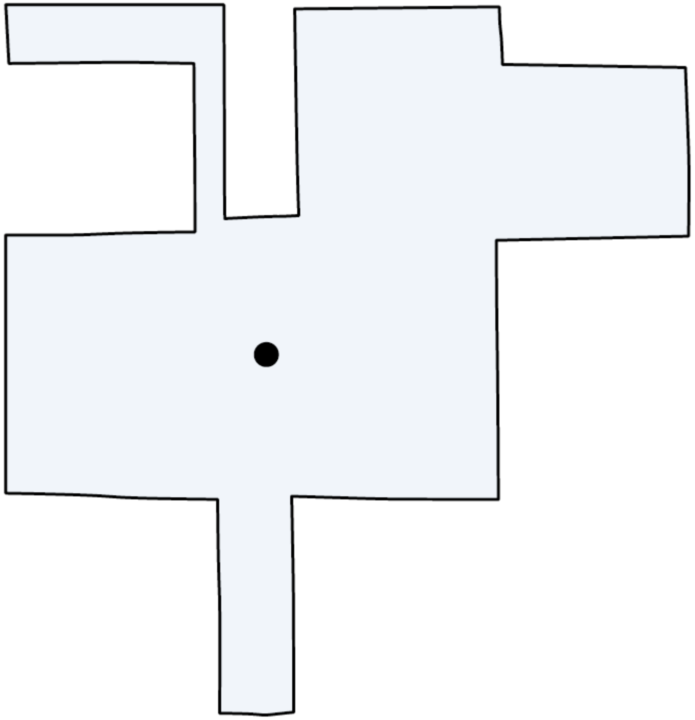
Our approach

- Instead of the classical approach in coding theory where errors are forming a Hamming sphere with radius t around the codewords



Our approach

- We want to form an arbitrary set of errors around the codewords
- We want also to increase the set of possible errors that the encryptor can introduce
- Still to be capable to decode efficiently and reliably



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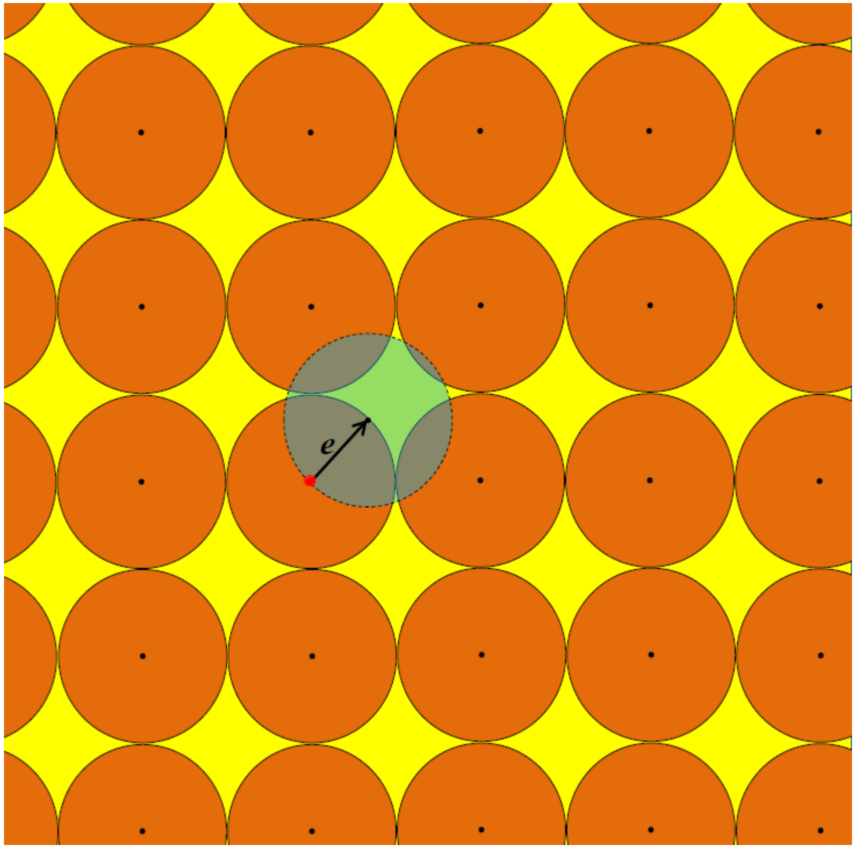
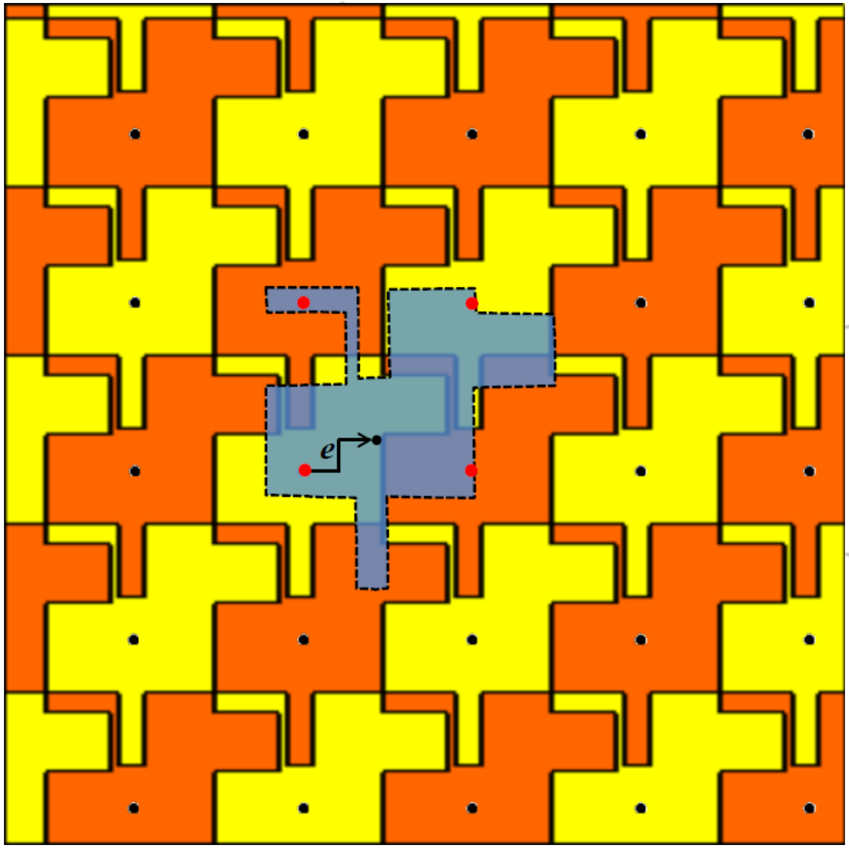
Our approach



A classical modeling of an error set around a code word with the Hamming sphere.



An artistic visualization of our idea with an arbitrary error set around the codewords (Escher's tessellations).

Our approach

- McEliece in the world of Escher
- Cryptology ePrint Archive: Report 2014/360
  http://eprint.iacr.org/2014/360

Some definitions and properties

Definition 1. *Let ℓ be a positive integer, and let $E_\ell \subset \mathbb{F}_2^\ell$ such that $|E_\ell| > 2^{l-1}$. We define the* density *of the set E_ℓ as:*

$$D(E_\ell) = |E_\ell|^{1/\ell}.$$

We will refer to the integer $\ell > 0$ as granulation *(when clear from context we will use just E).*

Proposition 1. *1. Let $E_{\ell_1} \subseteq \mathbb{F}_2^{\ell_1}$, $E_{\ell_2} \subseteq \mathbb{F}_2^{\ell_2}$, for some integers $\ell_1, \ell_2 > 0$. Let $D(E_{\ell_1}) = D(E_{\ell_2}) = \rho$. Then $D(E_{\ell_1} \times E_{\ell_2}) = \rho$.*

2. Let $E_{\ell,1}, E_{\ell,2}, \ldots, E_{\ell,m} \subseteq \mathbb{F}_2^\ell$, $\ell > 0$, and $D(E_{\ell,1}) = D(E_{\ell,2}) = \cdots = D(E_{\ell,m}) = \rho$. Then $D(E_{\ell,1} \times E_{\ell,2} \times \cdots \times E_{\ell,m}) = \rho$.

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Some definitions and properties

- Example

1. Let $E_2 = \{x \in \mathbb{F}_2^2 | wt(x) < 2\} = \{(0,0),(0,1),(1,0)\}$. Then $D(E_2) = |E_2|^{1/2} = 3^{1/2}$, and also $D(E_2^2) = |E_2^2|^{1/4} = 9^{1/4} = 3^{1/2}$ as well as $D(E_2^m) = 3^{1/2}$ for any positive integer m.
2. Let $E_{4,1} = \{x \in \mathbb{F}_2^4 | 2 \leq wt(x) \leq 3\}$. Then $D(E_{4,1}) = (\sum_{i=2}^{3} \binom{4}{i})^{1/4} = 10^{1/4}$, and also $D(E_{4,1}^m) = 10^{1/4}$ for any positive integer m. Note that the set $E_{4,2} = \{x \in \mathbb{F}_2^4 | wt(x) \leq 2\} \setminus \{(0,0,0,0)\}$ has also density $D(E_{4,2}) = 10^{1/4}$.
3. Let $E_4 = \{(0,1,0,0),(0,0,0,1),(0,1,0,1),(1,0,0,1),(0,0,1,0),(0,1,1,0),(1,0,1,0),(1,1,1,0),(0,1,1,1),(1,1,1,1)\}$. The values of E_4 are chosen without any particular rule in mind. Then $D(E_4) = |E_4|^{1/4} = 10^{1/4}$ as well as $D(E_4^m) = 10^{1/4}$ for any positive integer m.

Some definitions and properties

Proposition 3. *Let $\mathcal{C}$ be any binary (n,k) code and $E \subset \mathbb{F}_2^n$ be an error set of density ρ. Let $\mathbf{w}$ be any word of length n, $W_E = \{\mathbf{w}+\mathbf{e} | \mathbf{e} \in E\}$ and $\mathcal{C}_{W_E}$ denote the set of codewords in W_E. Then:*

1. *The expected number of codewords in W_E is $\rho^n 2^{k-n}$. The probability that $\mathcal{C}_{W_E}$ is an empty set is given by $\Pr[\mathcal{C}_{W_E} = \emptyset] \leq e^{-(\rho^n 2^{k-n+1}-1)^2/(\rho^n 2^{k-n+3})}$.*
2. *Suppose there exists a codeword $\mathbf{c} \in W_E$. Then the expected number of codewords in $W_E \setminus \{\mathbf{c}\}$ is approximately $\rho^n 2^{k-n}$ for large enough n and k. The probability that $\mathcal{C}_{W_E \setminus \{\mathbf{c}\}}$ has another element except $\mathbf{c}$ is estimated by $\Pr[|\mathcal{C}_{W_E \setminus \{\mathbf{c}\}}| \geq 1] \leq e^{-(1-\rho^n 2^{k-n})^2/(1+\rho^n 2^{k-n})}$.*

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1. Let $\mathcal{C}$ be a $(1280, 256)$ binary code. The code rate is 0.2. We consider an error set E of density $\rho = 3^{1/2}$. Let $\mathbf{c}$ be a codeword and $\mathbf{w} = \mathbf{c} + \mathbf{e}$ for some $\mathbf{e} \in E$. Then, from Proposition 3 the decoding list of the word $\mathbf{w}$ is of average length $1 + \rho^n 2^{k-n} =$ 1.00127. The probability that there is another element in the list except $\mathbf{c}$ is 0.37. Note that these parameters may be suitable for building an encryption scheme, since we can expect that the list has only one element.
2. Let $\mathcal{C}$ be a $(1208, 256)$ binary code. The code rate is 0.211921. We consider an error set E of density $\rho = 3^{1/2}$. Let $\mathbf{w}$ be a word of length n. Then, the decoding list of the word $\mathbf{w}$ is of average length 39.8733, and the probability that the list is empty is 2^{-28}. Such parameters are suitable for building a signature scheme, since with great confidence we can always expect to have a valid signature. Moreover, the number of valid signatures is relatively small.

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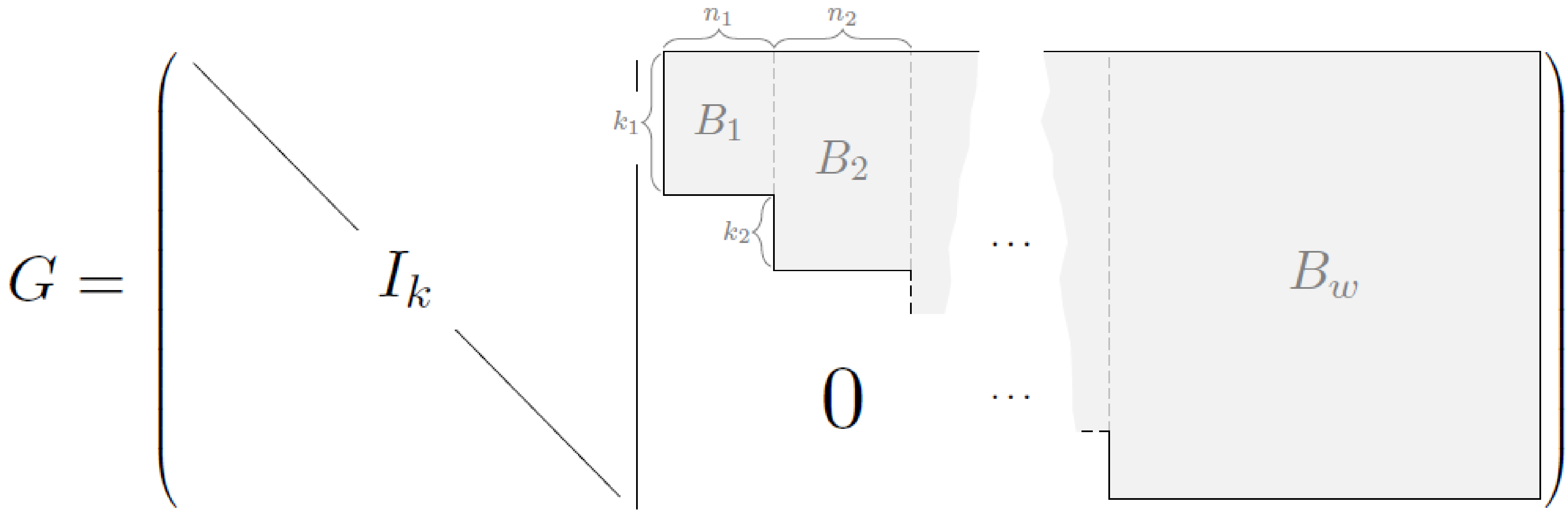
Some definitions and properties

- Example of Code where we can decode up to n/2 errors



Let $E_2 = \{x \in \mathbb{F}_2^2 | wt(x) < 2\} = \{(0,0),(0,1),(1,0)\}$. Then $D(E_2) = |E_2|^{1/2} = 3^{1/2}$, and also $D(E_2^2) = |E_2^2|^{1/4} = 9^{1/4} = 3^{1/2}$ as well as $D(E_2^m) = 3^{1/2}$ for any positive integer m.

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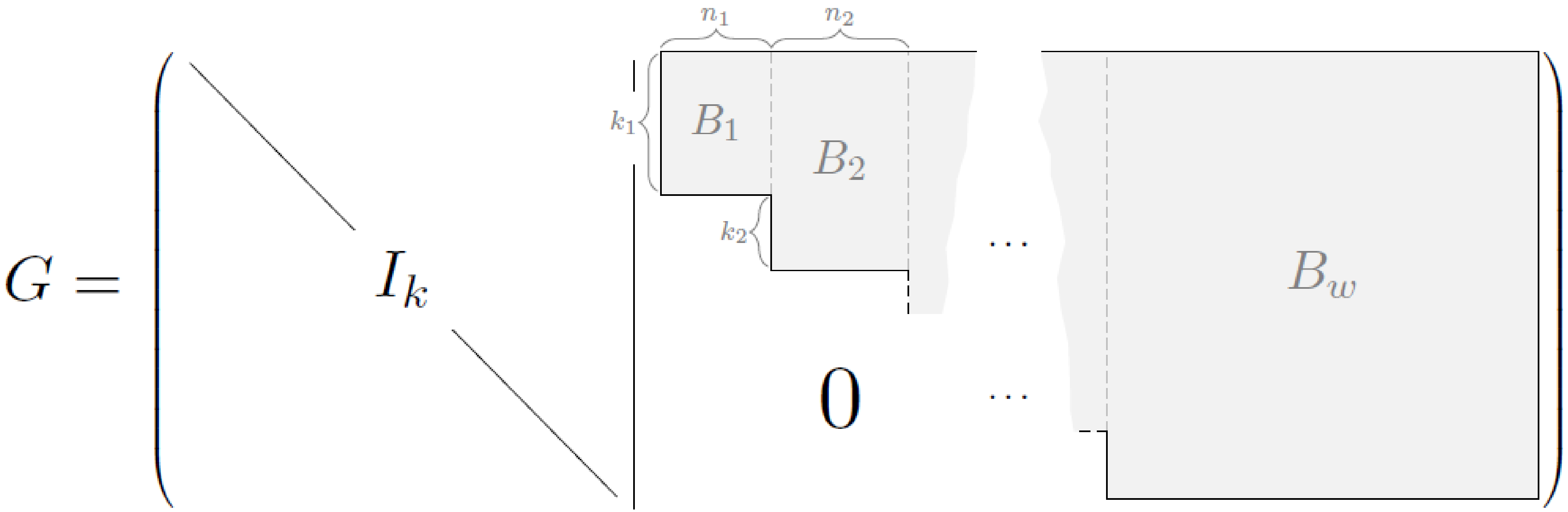

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Up to 1 error in every two bits AND up to n/2 errors

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Encryption

$$\mathbf{m} \in \mathbb{F}_2^k,$$

$$\mathbf{c} = \mathbf{m}G_{\text{pub}} + \mathbf{e} \in \mathbb{F}_2^n$$

$$\mathbf{e} \text{ (drawn from a specific error set } E^m)$$

Encryption

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The scheme looks similar as LWE schemes, but with different error set and powerful list decoding algorithm.

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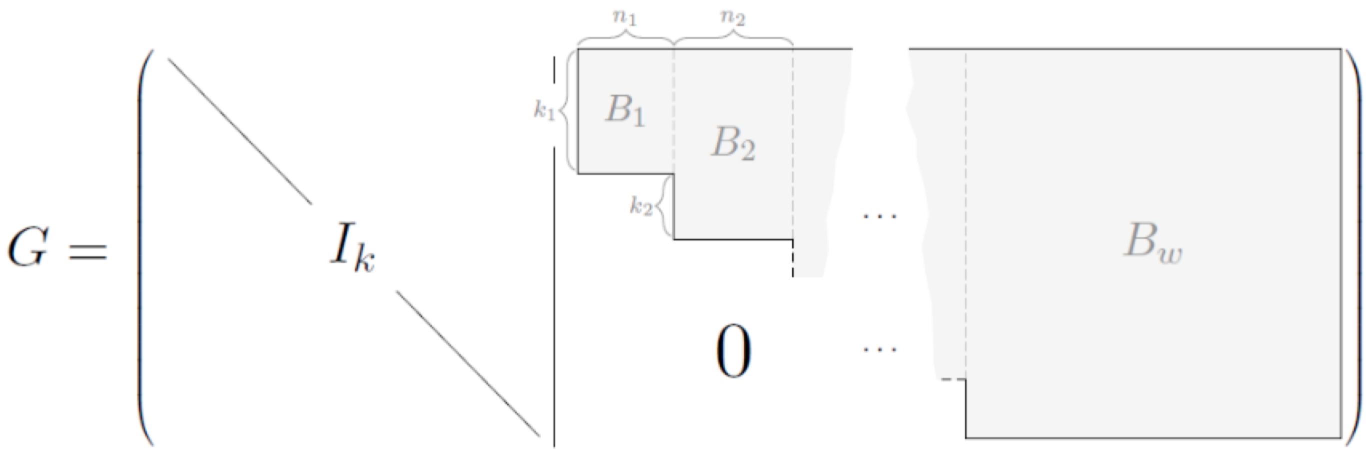
Decoding



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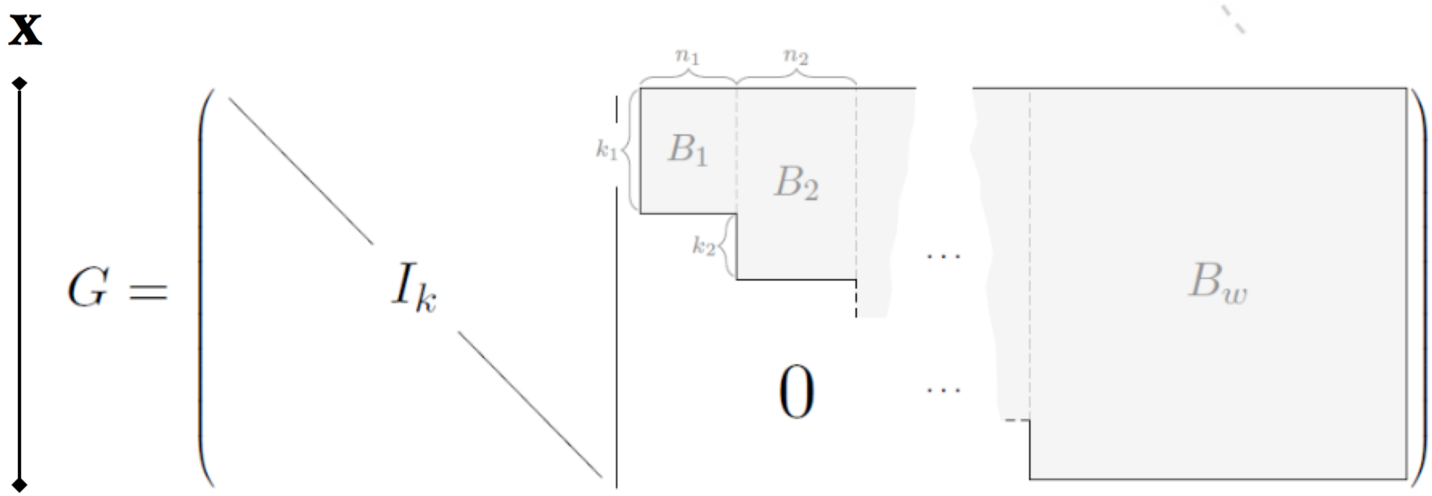

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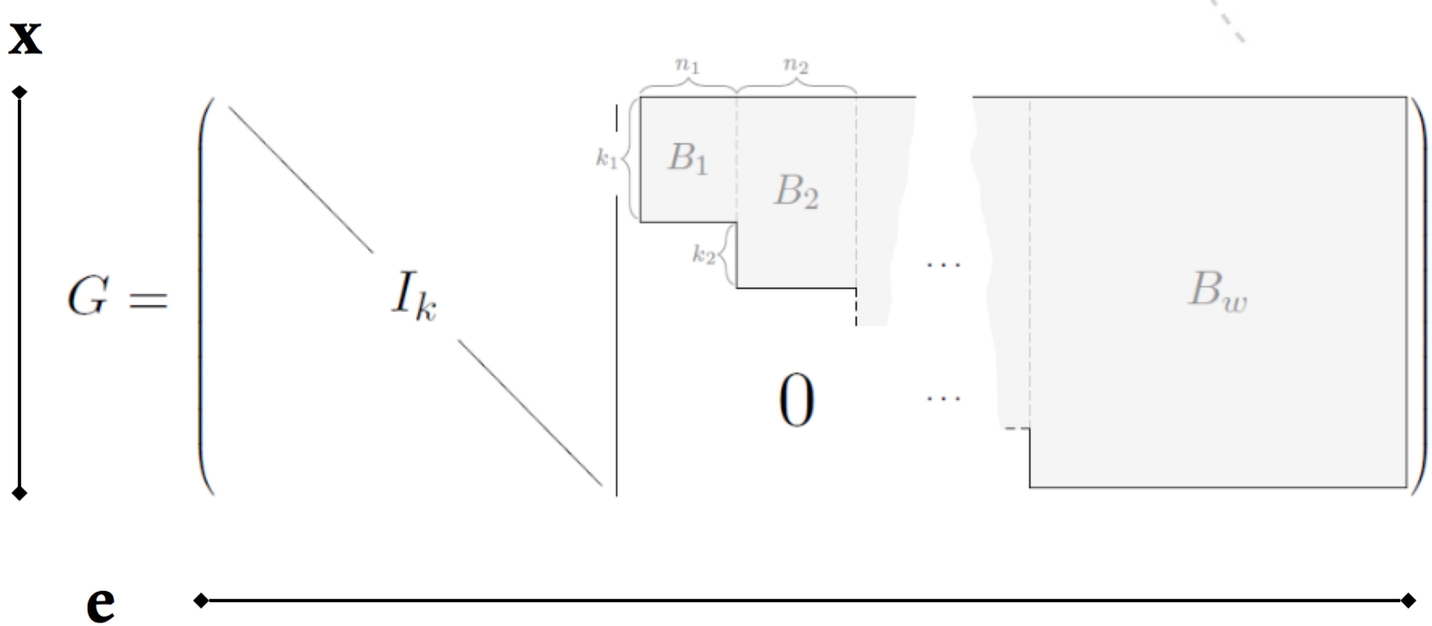

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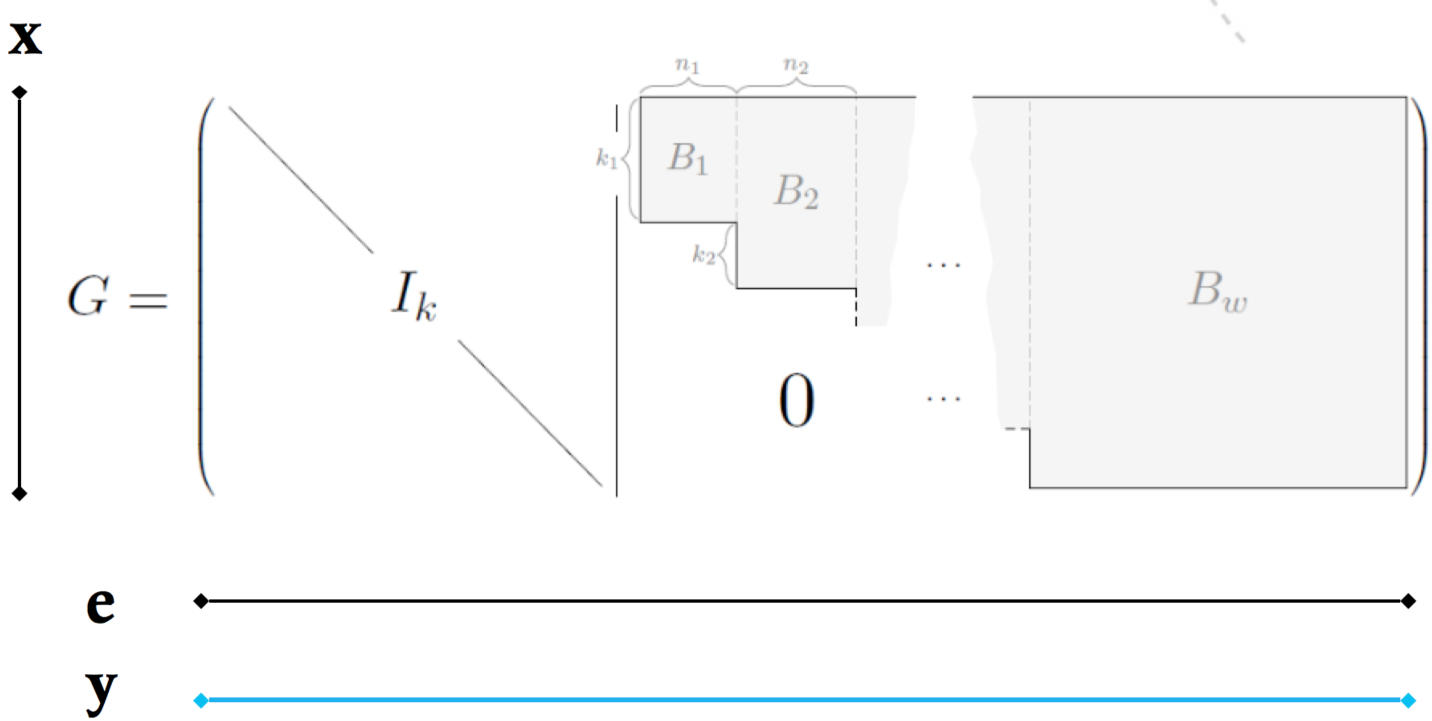

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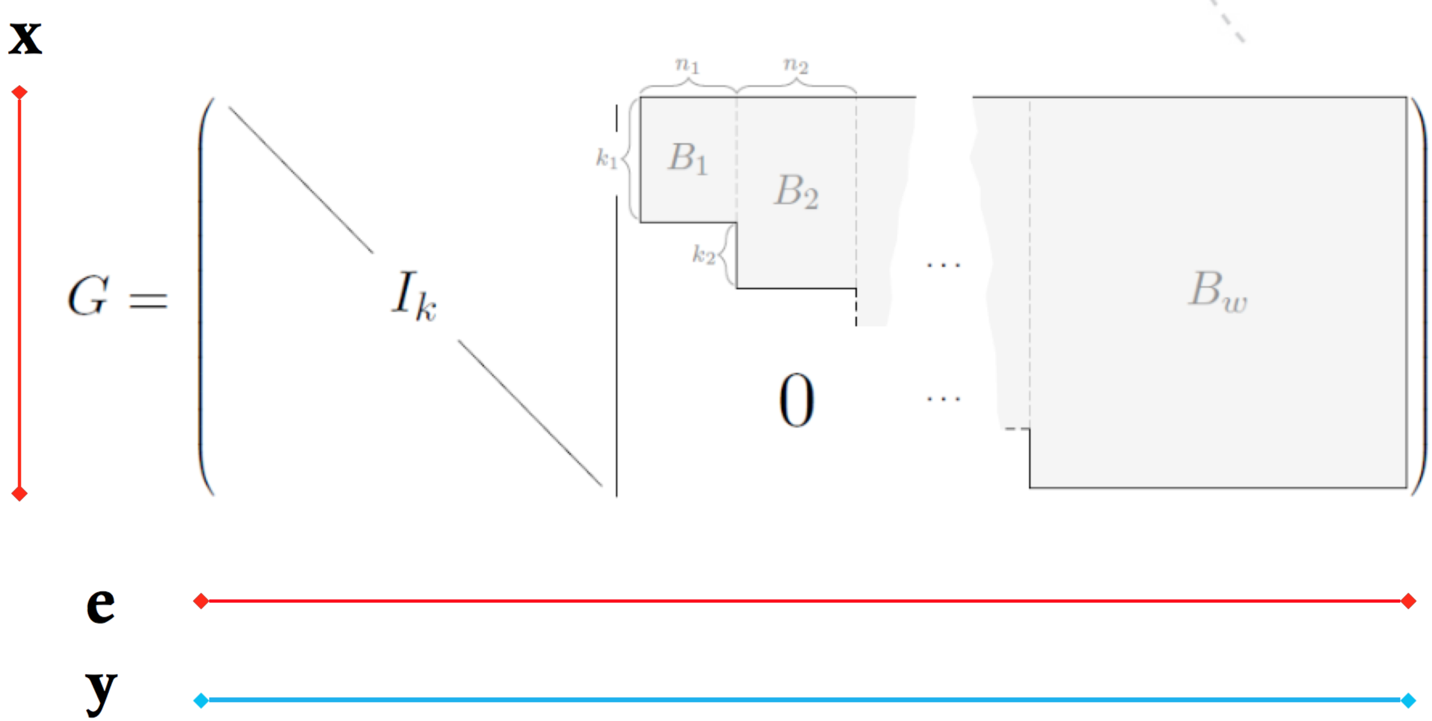

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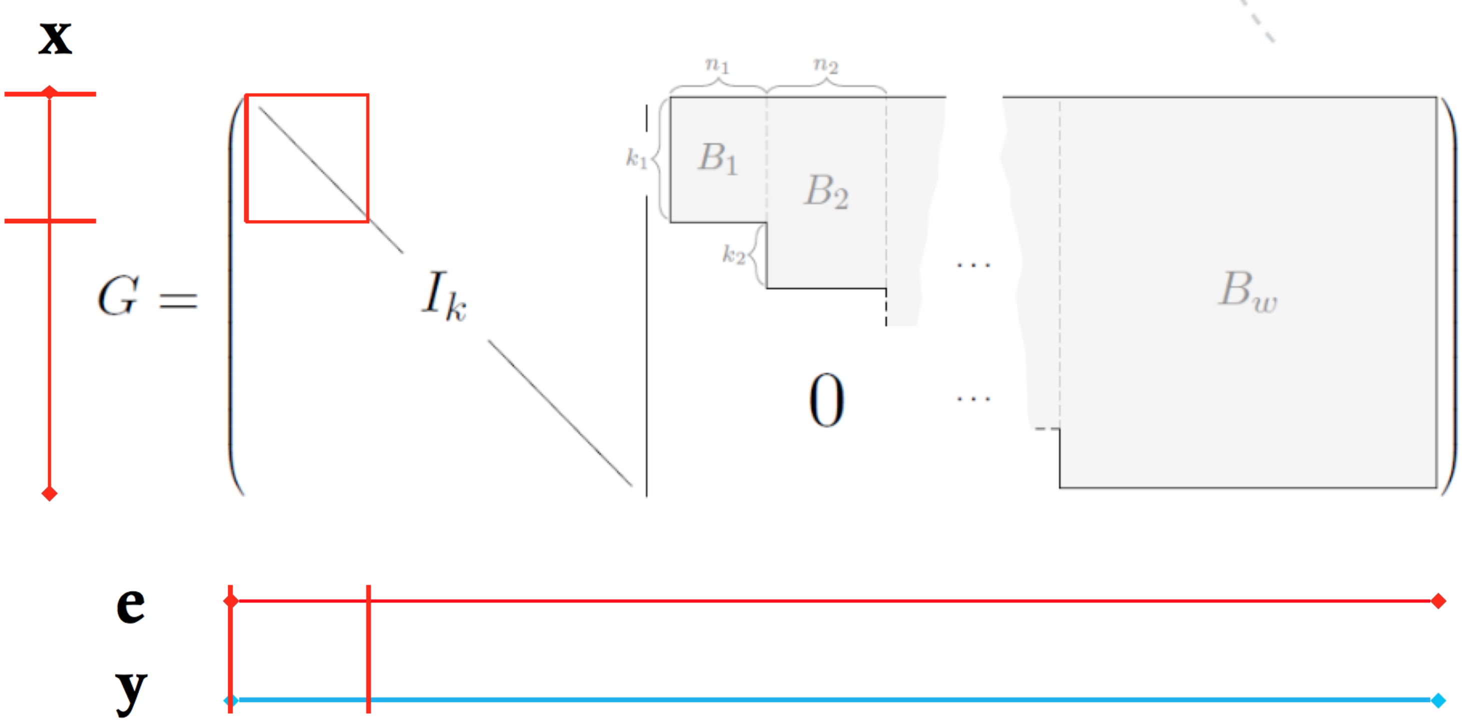

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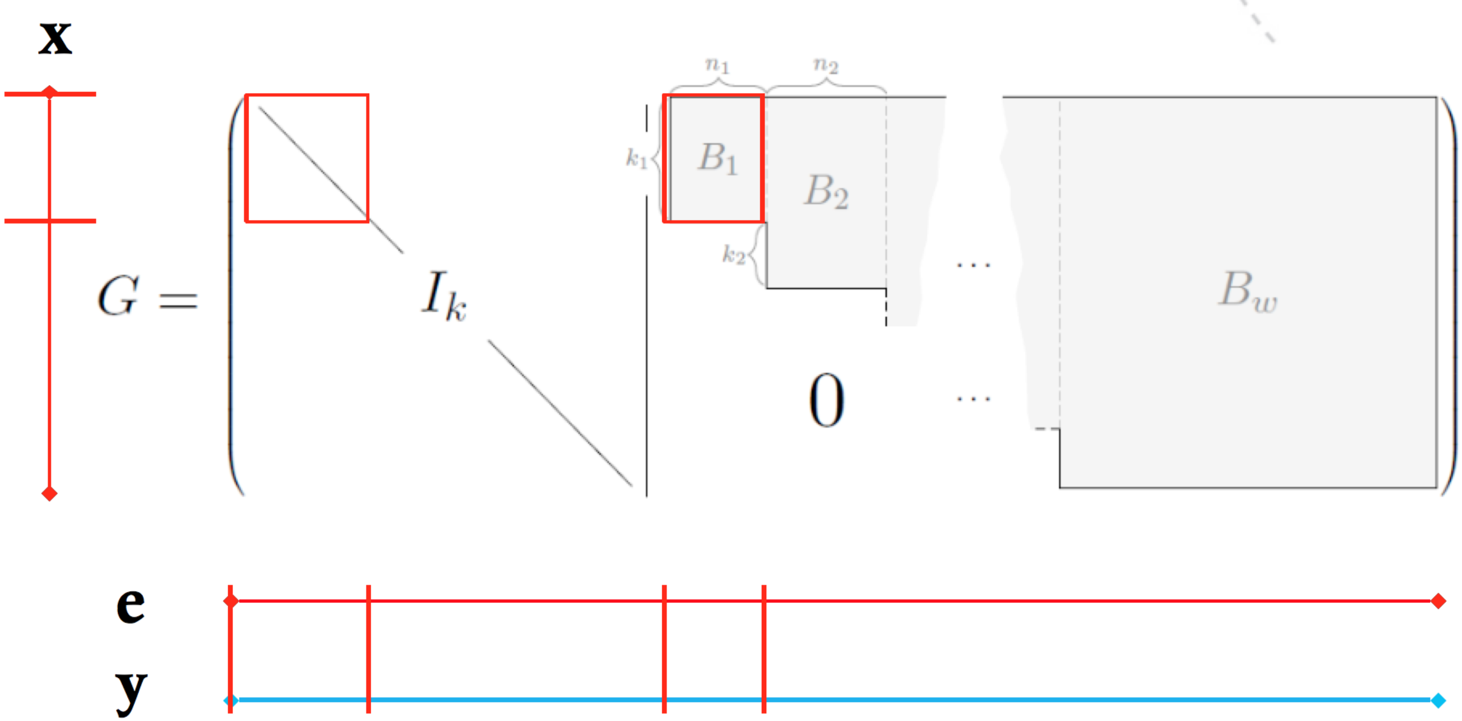

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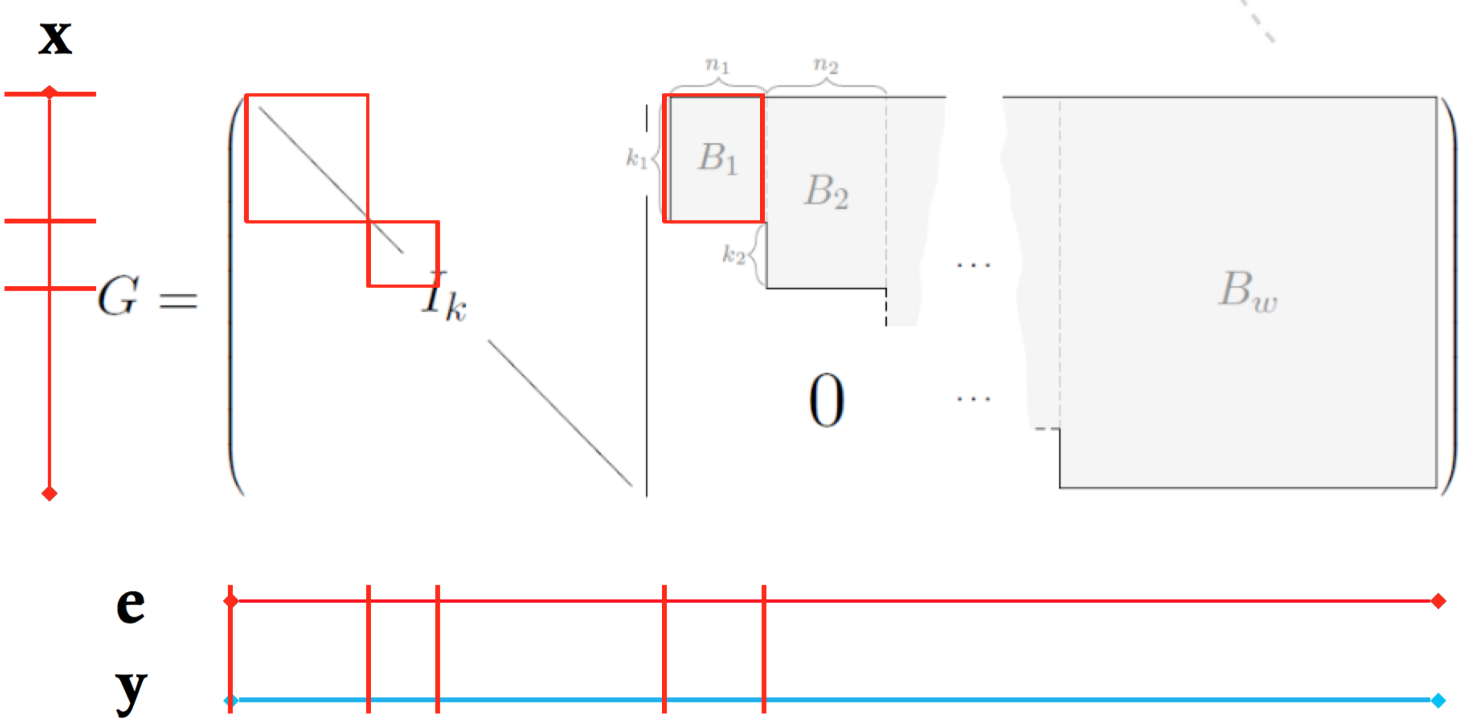

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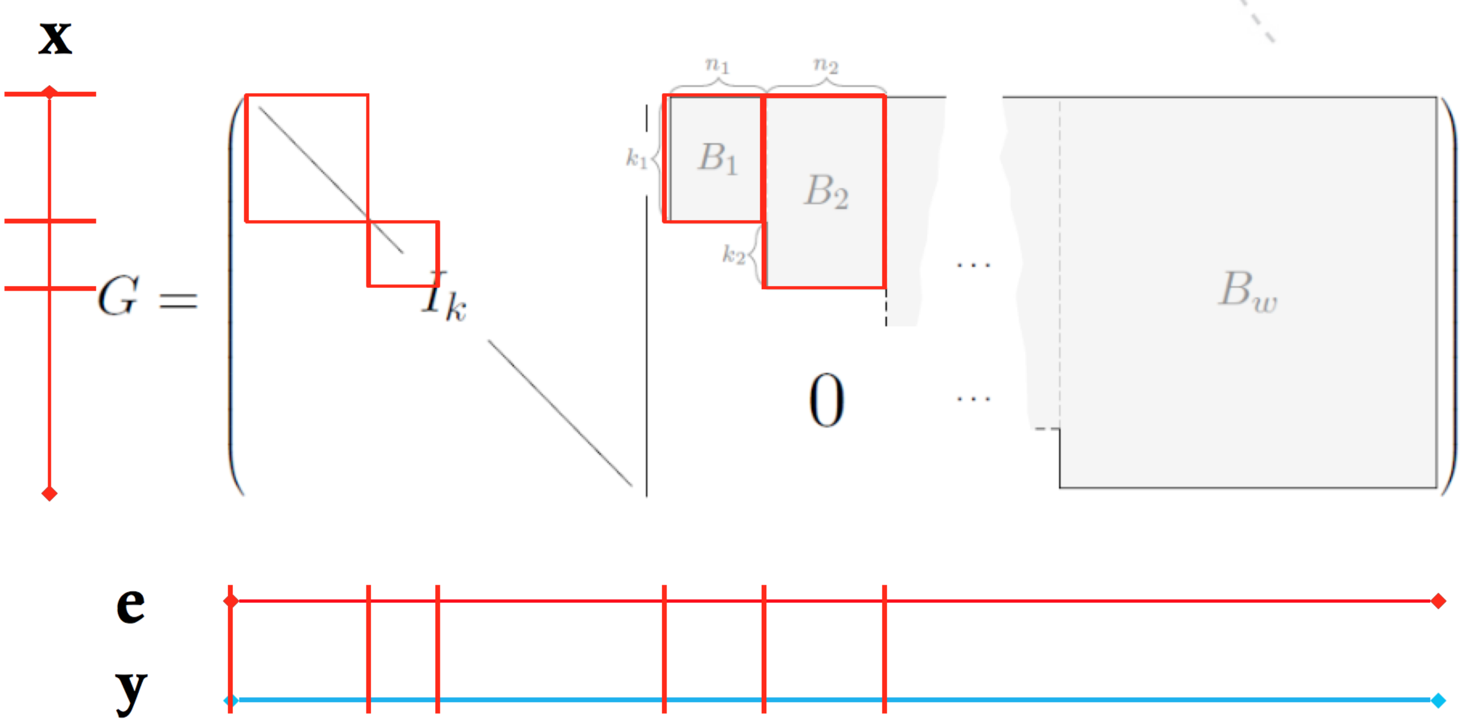

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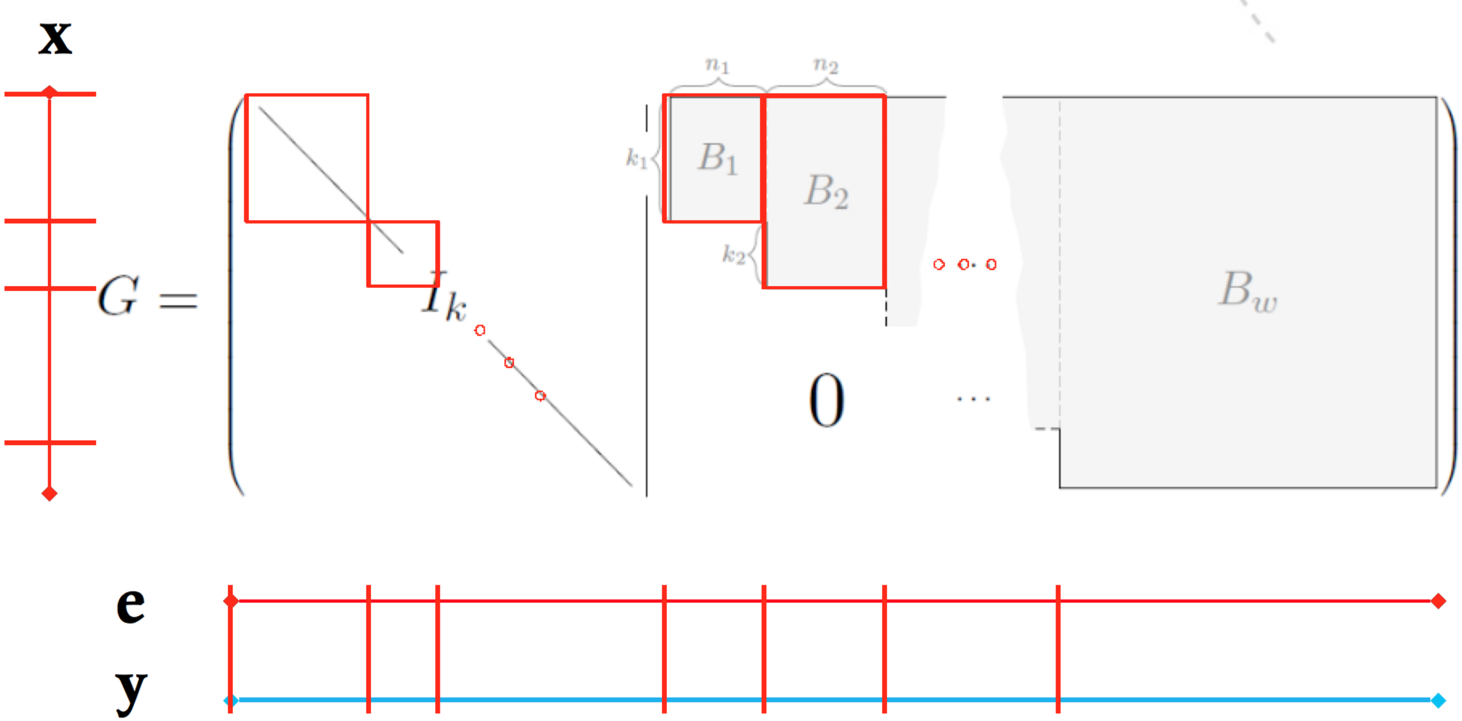

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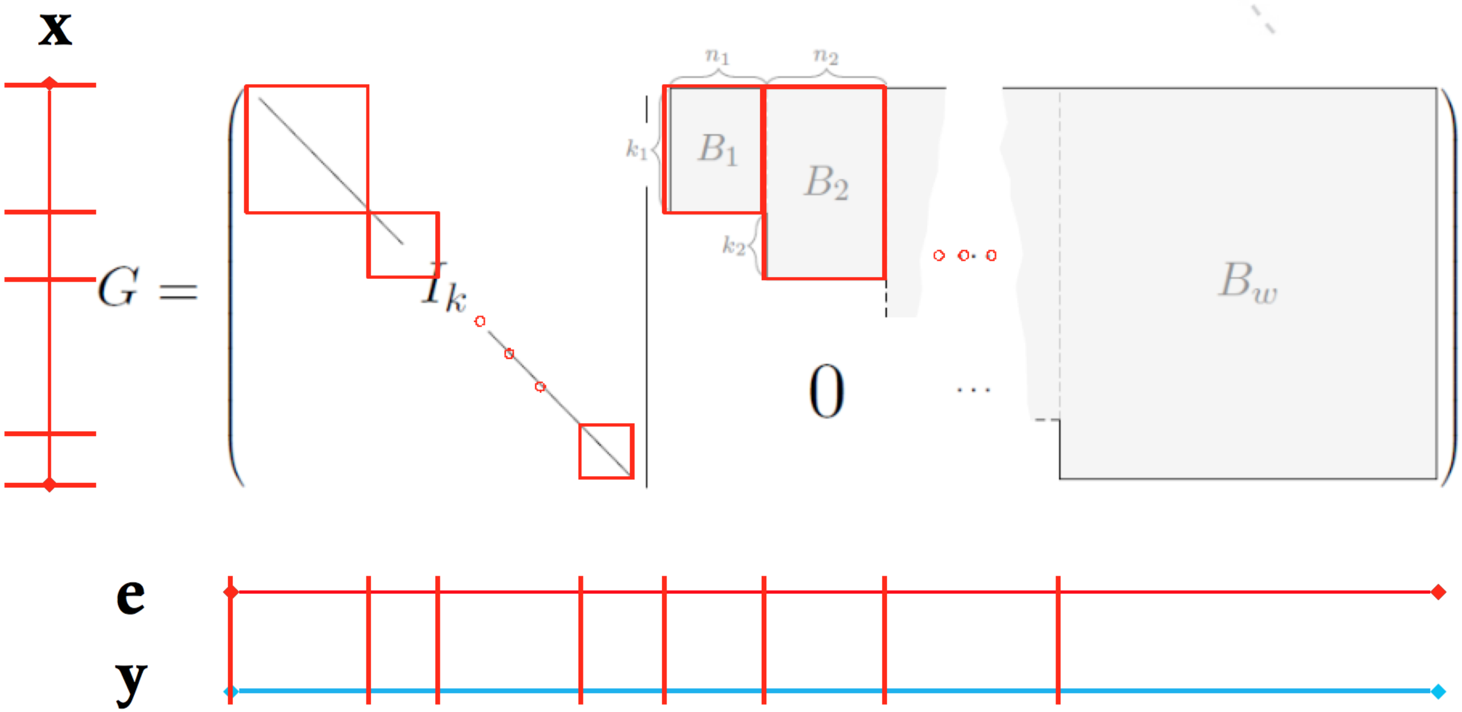

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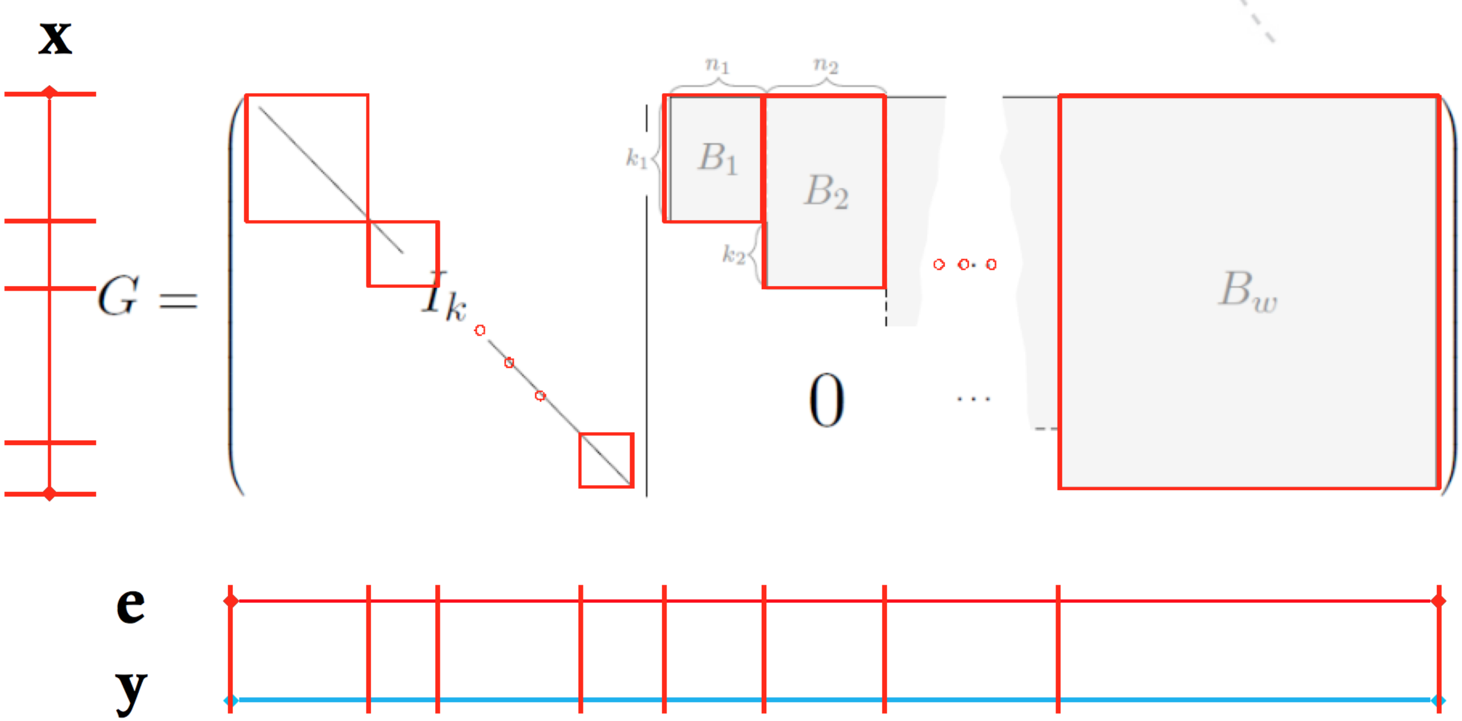

A nice property of the scheme

- **The scheme is both encryption and signature scheme**

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A nice property of the scheme

- **The scheme is both encryption and signature scheme**
- **Why the ordinary McEliece scheme is so hard to use it for digital signatures?**

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Recall …

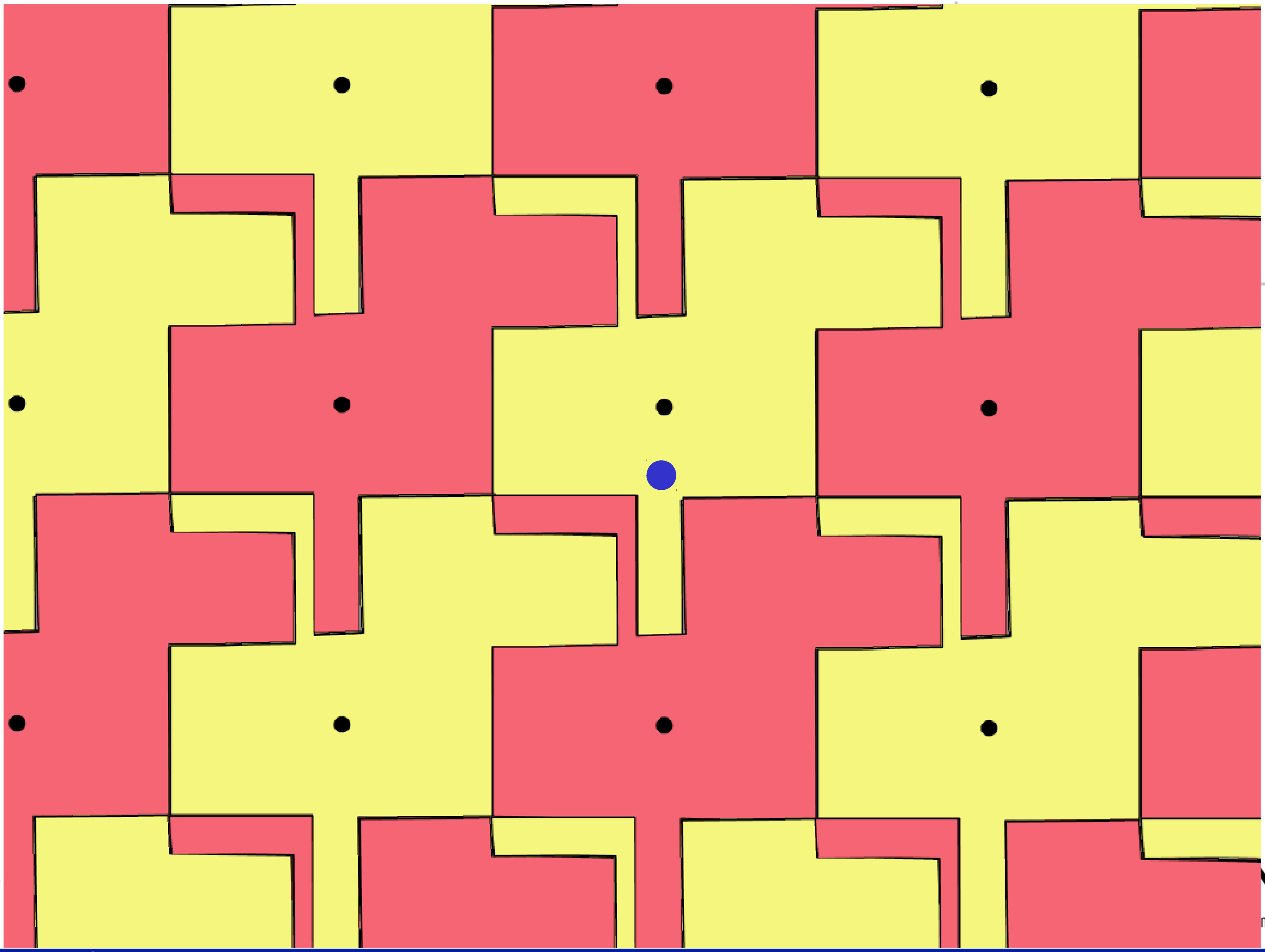
$t = d/2$

Recall …

The probability that a random syndrome lies in the "decodable areas" is **exponentially small**.

$t=d/2$

Recall …



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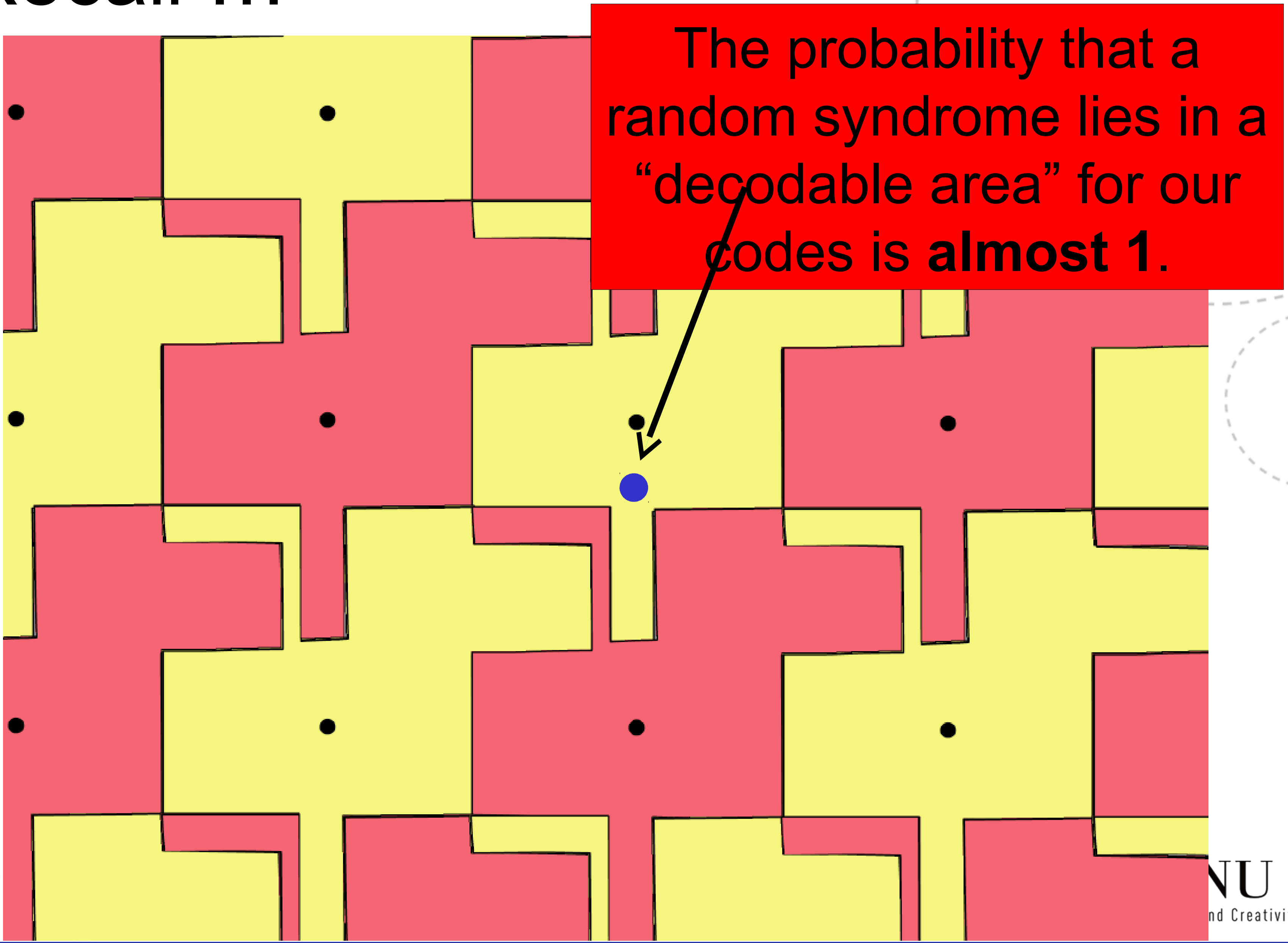
The probability that a random syndrome lies in a “decodable area” for our codes is **almost 1**.

A nice property of the scheme

Algorithm 3 Signing

Input: A value $\mathbf{h} \in \mathbb{F}_2^n$ to be signed. The private key S, G and P.

Output: A valid signature $\boldsymbol{\sigma} \in \mathbb{F}_2^k$, so that $\boldsymbol{\sigma} G_{\text{pub}} + \mathbf{h} \in E^m \subset \mathbb{F}_2^n$.

Procedure:

1. Compute $\mathbf{y} = \mathbf{h}P^{-1}$.
2. Decode $\mathbf{y}$ using Algorithm 1, to obtain a list L_w of (possibly several) valid decodings.
3. Select any element $\mathbf{x} \leftarrow L_w$ and compute $\boldsymbol{\sigma} = \mathbf{x}S^{-1}$.

Algorithm 4 Verification

Input: A pair $(\mathbf{h}, \boldsymbol{\sigma}) \in \mathbb{F}_2^n \times \mathbb{F}_2^k$, and the public key G_{pub}.

Output:

$$\mathrm{Ver}(\mathbf{h}, \boldsymbol{\sigma}) = \begin{cases} \text{Accept}, & \text{if } \boldsymbol{\sigma} G_{\text{pub}} + \mathbf{h} \overset{?}{\in} E^m \subset \mathbb{F}_2^n. \\ \text{Reject}, & \text{otherwise.} \end{cases}$$

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A nice property of our approach

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Recall (we can control the number of elements in the decoding list):

Let $\mathcal{C}$ be a $(1208, 256)$ binary code. The code rate is 0.211921. We consider an error set E of density $\rho = 3^{1/2}$. Let $\mathbf{w}$ be a word of length n. Then, the decoding list of the word $\mathbf{w}$ is of average length $\boxed{39.8733,}$ and the probability that the list is empty is 2^{-28}. Such parameters are suitable for building a signature scheme, since with great confidence we can always expect to have a valid signature. Moreover, the number of valid signatures is relatively small.

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$$\mathrm{Ver}(\mathbf{h}, \boldsymbol{\sigma}) = \begin{cases} \text{Accept}, & \text{if } \boldsymbol{\sigma} G_{\text{pub}} + \mathbf{h} \overset{?}{\in} E^m \subset \mathbb{F}_2^n. \\ \text{Reject}, & \text{otherwise} \end{cases}$$

Recall (we can control the number of elements in the decoding list):

Let $\mathcal{C}$ be a $(1208, 256)$ binary code. The code rate is 0.211921. We consider an error set E of density $\rho = 3^{1/2}$. Let $\mathbf{w}$ be a word of length n. Then, the decoding list of the word $\mathbf{w}$ is of average length 39.8733, and the probability that the list is empty is 2^{-28}. Such parameters are suitable for building a signature scheme, since with great confidence we can always expect to have a valid signature. Moreover, the number of valid signatures is relatively small.

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Security analysis

- **Four types of attacks analyzed**

1. **Information Set Decoding**

   for Error Sets of a Given Density

2. **Modeling ISD using Polynomial System Solving**

   Groebner bases apprach

3. **Rank Attacks**
4. **Cheap distinguishers (equivalent keys finders)**

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Security analysis

1. Information Set Decoding

for Error Sets of a Given Density

Theorem 1. *The probability of success of one iteration and the cost of one iteration of the Lee-Brickell variant, Stern variant, Finiasz-Sendrier variant, Bernstein-Lange-Peters variant, May-Meurer-Thomae variant and Becker-Joux-May-Meurer variant adapted to error sets of density ρ are given in Table 1.*

Variant	ρPr_{VAR}	$\rho Cost_{VAR} - Cost_{Gauss}$
LB	$\binom{k/\ell}{p}\frac{(\rho^\ell-1)^p}{\rho^k}$	$\binom{k/\ell}{p}(\rho^\ell-1)^p pn$
ST	$\binom{k/2\ell}{p}^2\frac{(\rho^\ell-1)^{2p}}{\rho^{k+\lambda}}$	$2\lambda pL + 2pn\frac{L^2}{2^\lambda}, \quad L = \binom{k/2\ell}{p}(\rho^\ell-1)^p$
FS	$\binom{(k+\lambda)/2\ell}{p}^2\frac{(\rho^\ell-1)^{2p}}{\rho^{k+\lambda}}$	$2\lambda pL + 2pn\frac{L^2}{2^\lambda}, \quad L = \binom{(k+\lambda)/2\ell}{p}(\rho^\ell-1)^p$
BLP	$\binom{k/2\ell}{p}^2\binom{\lambda_1/\ell}{q}\binom{\lambda_2/\ell}{q} \cdot \frac{(\rho^\ell-1)^{2p+2q}}{\rho^{k+\lambda_1+\lambda_2}}$	$\binom{k/2\ell}{p}(\rho^\ell-1)^p 2(\lambda_1+\lambda_2)p + \binom{k/2\ell}{p}\left(\binom{\lambda_1/\ell}{q}+\binom{\lambda_2/\ell}{q}\right)(\rho^\ell-1)^{p+q}(\lambda_1+\lambda_2)q$ $+ \frac{\binom{k/2\ell}{p}^2\binom{\lambda_1/\ell}{q}\binom{\lambda_2/\ell}{q}(\rho^\ell-1)^{2p+2q}}{2^{\lambda_1+\lambda_2}}2(p+q)n$
MMT	$\binom{(k+\lambda)/2\ell}{p}^2\frac{(\rho^\ell-1)^{2p}}{\rho^{k+\lambda}}$	$2\lambda_2 pL + (2n+\lambda-\lambda_2)p\frac{L^2}{2^{\lambda_2}} + pn\frac{L^4}{2^{\lambda+\lambda_2}}, \quad L = \binom{(k+\lambda)/2\ell}{p/2}(\rho^\ell-1)^{p/2}$
$BJMM$	$\binom{(k+\lambda)/\ell}{p}\frac{(\rho^\ell-1)^p}{\rho^{k+\lambda}}$	$4Pr_{coll}^{-4}p_2\left(L_3\log_2 R_2 + n\frac{L_3^2}{R_2}\right) + 2n\left(p_1\frac{L_2^2R_2}{R_1} + p\frac{L_1^2R_1}{2^\lambda}\right),$ $Pr_{coll} = \binom{(k+\lambda)/2\ell}{p_2/2}^2\binom{(k+\lambda)/\ell}{p_2}^{-1}, \; p_i = \frac{p_{i-1}}{2}+\epsilon_i, \; i=1,2, p_0=p,$ $L_i = \binom{(k+\lambda)/2\ell}{p_i}(\rho^\ell-1)^{p_i}, i=1,2, \quad L_3 = \binom{(k+\lambda)/2\ell}{p_2/2}(\rho^\ell-1)^{p_2/2},$ $R_i = \binom{p_{i-1}}{p_{i-1}/2}\binom{(k+\lambda)/\ell-p_{i-1}}{\epsilon_i}(\rho^\ell-1)^{\epsilon_i}, \; i=1,2, p_0=p$

Variant	ρPr_{VAR}	$\rho Cost_{VAR} - Cost_{Gauss}$
LB	$\binom{k/\ell}{p}\frac{(\rho^{\ell}-1)^{p}}{\rho^{k}}$	$\binom{k/\ell}{p}(\rho^{\ell}-1)^{p}pn$
ST	$\binom{k/2\ell}{p}^{2}\frac{(\rho^{\ell}-1)^{2p}}{\rho^{k+\lambda}}$	$2\lambda pL + 2pn\frac{L^2}{2^{\lambda}}, \quad L = \binom{k/2\ell}{p}(\rho^{\ell}-1)^{p}$
MMT	$\binom{(k+\lambda)/2\ell}{p}^{2}\frac{(\rho^{\ell}-1)^{2p}}{\rho^{k+\lambda}}$	$2\lambda_2 pL + (2n+\lambda-\lambda_2)p\frac{L^2}{2^{\lambda_2}} + pn\frac{L^4}{2^{\lambda+\lambda_2}}, \quad L = \binom{(k+\lambda)/2\ell}{p/2}(\rho^{\ell}-1)^{p/2}$
$BJMM$	$\binom{(k+\lambda)/\ell}{p}\frac{(\rho^{\ell}-1)^{p}}{\rho^{k+\lambda}}$	$4Pr_{coll}^{-4}p_2\left(L_3\log_2 R_2 + n\frac{L_3^2}{R_2}\right) + 2n\left(p_1\frac{L_2^2R_2}{R_1} + p\frac{L_1^2R_1}{2^{\lambda}}\right),$ $Pr_{coll} = \binom{(k+\lambda)/2\ell}{p_2/2}^{2}\binom{(k+\lambda)/\ell}{p_2}^{-1}, \ p_i = \frac{p_{i-1}}{2}+\epsilon_i, \ i=1,2, p_0=p,$ $L_i = \binom{(k+\lambda)/2\ell}{p_i}(\rho^{\ell}-1)^{p_i}, i=1,2, \quad L_3 = \binom{(k+\lambda)/2\ell}{p_2/2}(\rho^{\ell}-1)^{p_2/2},$ $R_i = \binom{p_{i-1}}{p_{i-1}/2}\binom{(k+\lambda)/\ell-p_{i-1}}{\epsilon_i}(\rho^{\ell}-1)^{\epsilon_i}, \ i=1,2, p_0=p$

Variant	LB	ST	FS	BLP	MMT	$BJMM$
$k=256$	2^{212}	2^{197}	2^{186}	2^{186}	2^{146}	2^{123}
$k=512$	2^{416}	2^{381}	2^{356}	2^{356}	2^{279}	2^{226}

Security analysis

2. Modeling ISD using Polynomial System Solving

Groebner bases apprach

Example 1.

Let the error set be E_2={(0,0), (0,1), (1,0)}. This set is completely described by the following **quadratic** equation:

$$e_1 e_2 = 0,$$

i.e. the solutions of that equation coincides with the error set.

Example 2.

Let the error set be E_4={(0,1,0,0), (0,0,0,1), (0,1,0,1), (1,0,0,1), (0,0,1,0), (0,1,1,0), (1,0,1,0), (1,1,1,0), (0,1,1,1), (1,1,1,1)}. This set is completely described by the following **cubic** equation:

$$e_2 + e_3 + e_4 + e_1 e_2 + e_2 e_3 + e_2 e_4 + e_1 e_2 e_3 = 1.$$

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2. Modeling ISD using Polynomial System Solving

Groebner bases apprach

Given a public generator matrix G_{pub} and a ciphertext $\mathbf{c}$, we can form n linear equations

$$\mathbf{x}G_{\text{pub}} + \mathbf{y} = \mathbf{c},$$

where $\mathbf{x}$ denotes the k unknown bits of the message, and $\mathbf{y}$ is the n-bit unknown error. Clearly, we don't have enough equations to find the correct solution efficiently. However, from the known structure of the error vector we can derive additional equations of higher degree that describe exactly the error set. If we denote these equations as $P(\mathbf{y}) = 0$, then a solution of the system

$$\begin{aligned} \mathbf{x}G_{\text{pub}} + \mathbf{y} &= \mathbf{c} \\ P(\mathbf{y}) &= 0 \end{aligned} \tag{13}$$

will give the same solution for the message and the error vector as the decoding algorithm with the knowledge of the private key.

We emphasize that any error set can be described by a system of equations, including the set of errors of a bounded weight used in the McEliece system. The efficiency of this approach strongly depends on the error structure.

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2. Modeling ISD using Polynomial System Solving

Groebner bases apprach

Furthermore, it is possible to introduce an optimization parameter in the form of a guess of some of the errors, or a guess of linear equations for the errors. In what follows we present the modeling of an error set of density $\rho = 3^{1/2}$ and granulation $\ell = 2$.

Let E_ℓ be an error set of density $\rho = 3^{1/2}$ and granulation $\ell = 2$. Without loss of generality, we can assume that $E_\ell = \{(00), (01), (10)\}$. Let $(e_1, e_2) \in E_\ell$. Then, the equation $e_1 e_2 = 0$ describes completely the error set E_ℓ. Hence, the system ($\boxed{13}$) turns into:

$$\begin{aligned} (x_1, \ldots, x_k)G_{\text{pub}} + (y_1, \ldots, y_n) &= \mathbf{c} \\ y_1 y_2 &= 0 \\ &\ldots \\ y_{n-1} y_n &= 0 \end{aligned}$$

The system can be easily transformed to the following form:

$$\begin{aligned} A_1(x_1, \ldots, x_k) A_2(x_1, \ldots, x_k) &= 0 \\ &\ldots \\ A_{n-1}(x_1, \ldots, x_k) A_n(x_1, \ldots, x_k) &= 0 \end{aligned} \tag{14}$$

where A_i are some affine expressions in the variables $x_1, \ldots, x_k$.

Security analysis

2. Modeling ISD usin

Groebner bases app

Given a concrete encryption **c**, we model the algebraic system to solve it by Groebner bases as a MQ system where the unknown variables are from the **message** and the **error**.

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We can look at this scheme as the first public key encryption and signature **multivariate (quadratic, cubic, ...)** scheme that is not given in a form

S · P · T

The system can

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$$\ldots \tag{14}$$
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where A_i are some affine expressions in the variables $x_1, \ldots, x_k$.

Security analysis

2. Modeling ISD using Polynomial System Solving

Groebner bases apprach

We can introduce an optimization parameter p as follows. Suppose we have made a correct guess that the equation $y_{2t-1} + y_{2t} = b_t$, $b_t \in \{0,1\}$ holds for p pairs (y_{2t-1}, y_{2t}) of coordinates of the error vector. Adding these p new equations to the system reduces the complexity of solving it. Note that it is enough to correctly guess k equation to obtain a full system of k unknowns. The probability of making the correct guess is $Pr = (2/3)^p$. Under the natural constrain $0 \le p \le k$, we can roughly estimate the complexity to

$$Comp = (3/2)^p \cdot \left(\binom{k-p}{Dreg_{k-p}} + p \right)^{\omega}$$

Security analysis

2. Modeling ISD using Polynomial System Solving

Groebner bases apprach

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$$Comp = (3/2)^p \cdot \left(\binom{k-p}{Dreg_{k-p}} + p \right)^{\omega}$$

where $Dreg_{k-p}$ denotes the degree of regularity of a system of $k-p$ variables of the form (14).

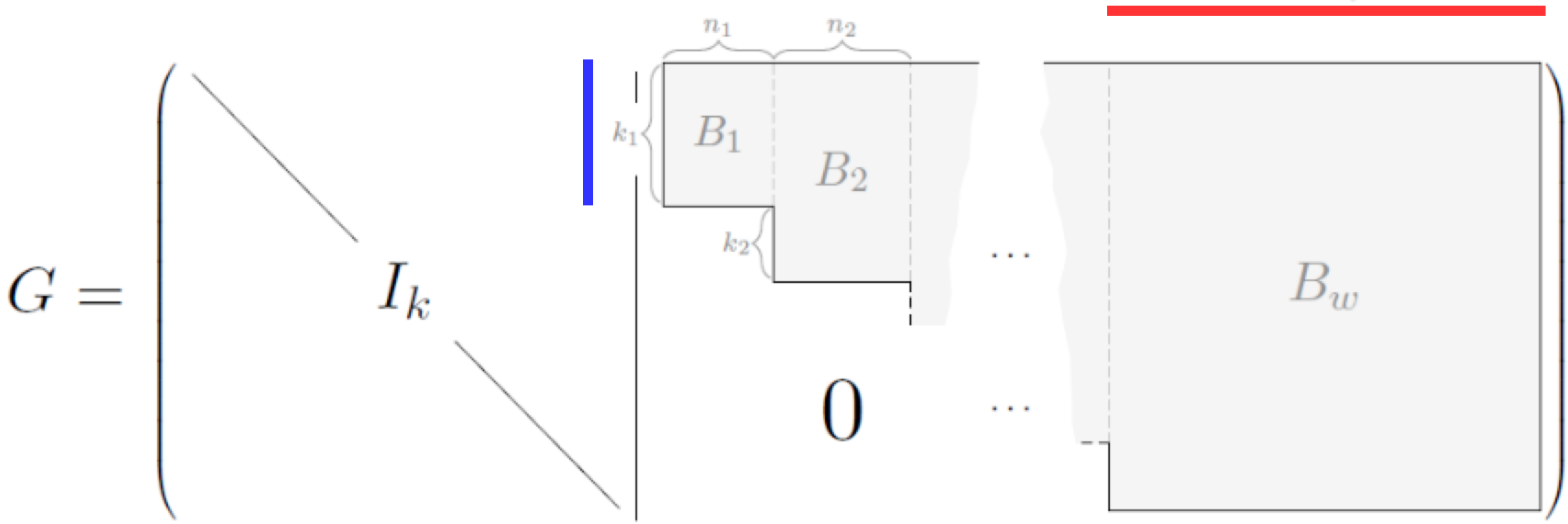
We performed some experiments using the F_4 algorithm [19] implemented in MAGMA [34], and based on rather conservative projections of the degree of regularity, we give the following table with a rough estimate of the lower bound of the complexity.

Table 3. Estimated complexity of solving ρISD using the F_4 algorithm for $\ell = 2$, $\rho = 3^{1/2}$.

k	Complexity
128	2^{84}
256	2^{152}
512	2^{237}

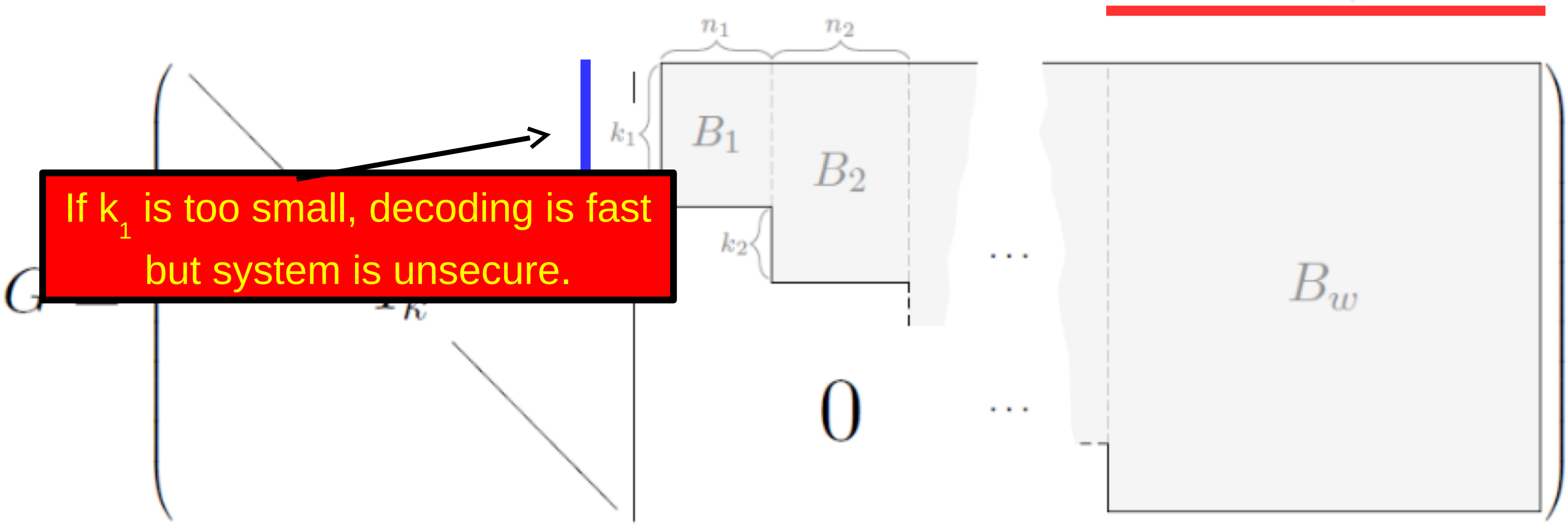
Security analysis

3. Rank Attacks (on the code and on the dual code)



Security analysis

3. **Rank Attacks (on the code and on the dual code)**



Security analysis

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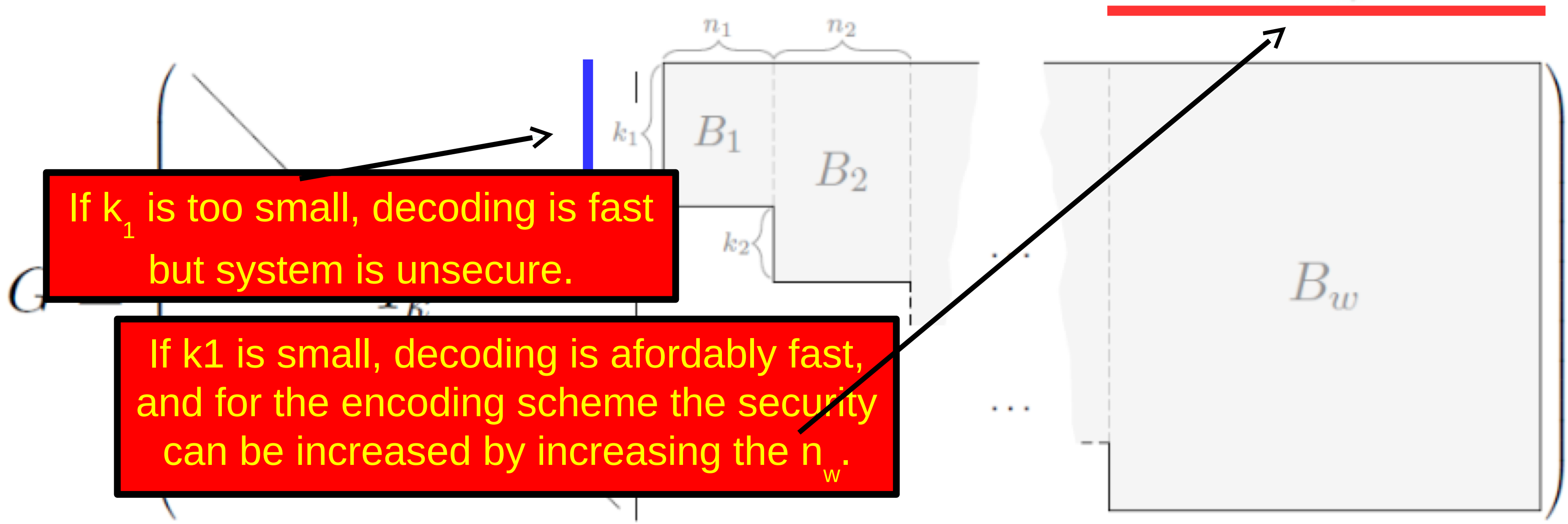

If k_1 is too small, decoding is fast but system is unsecure.

If k1 is small, decoding is afordably fast, and for the encoding scheme the security can be increased by increasing the n_w.

Security analysis

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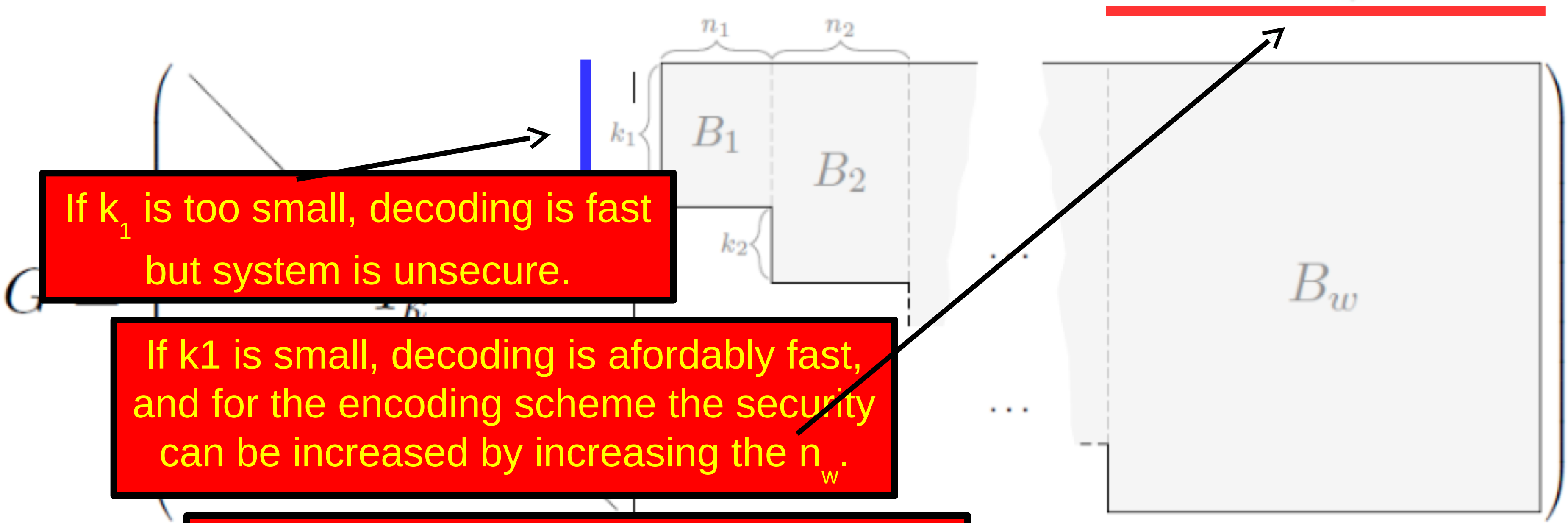

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But n_w can not be too big, because Groebner attacks will break the system.

Security analysis

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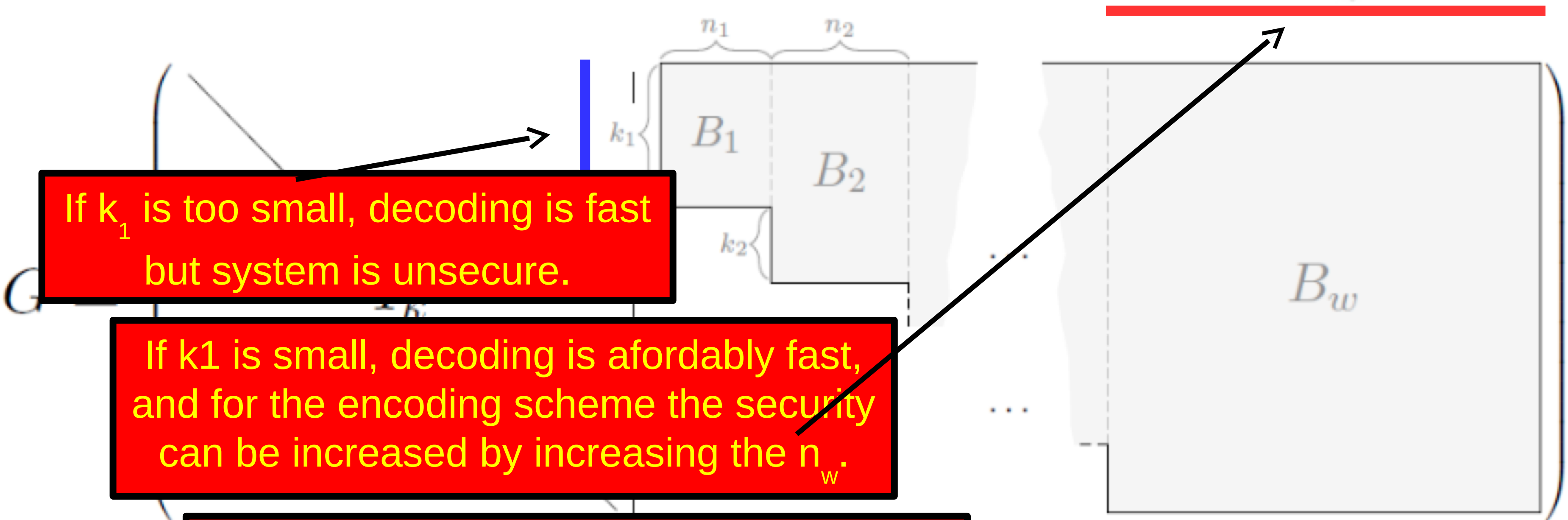

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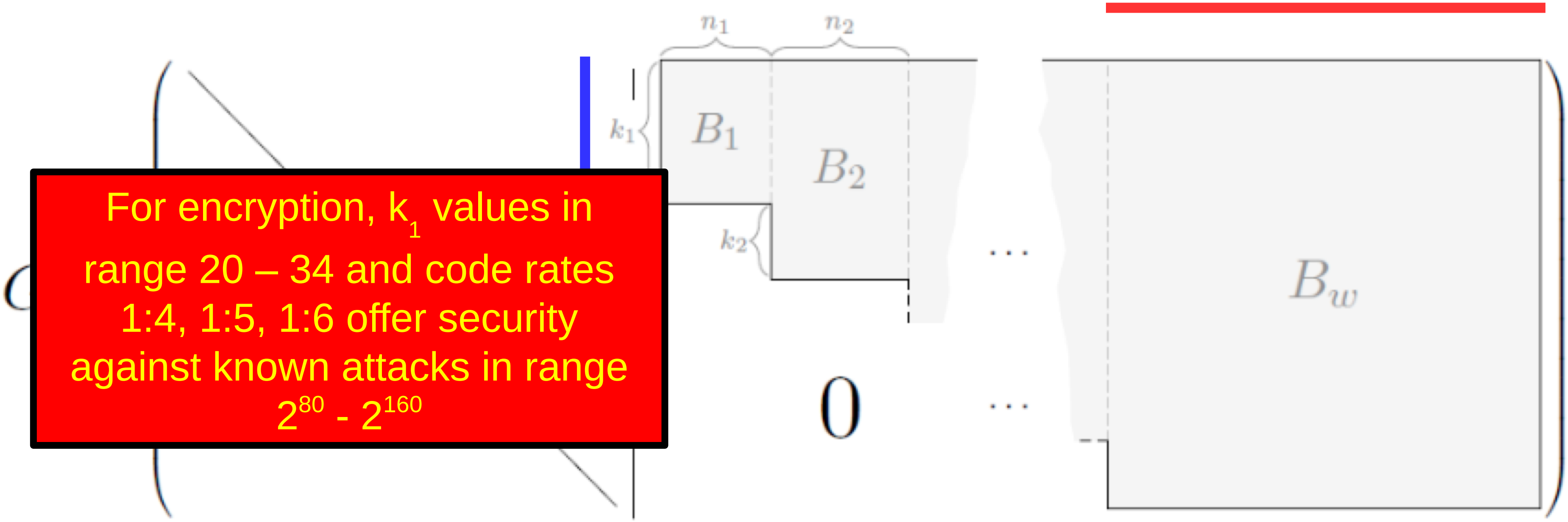
If k1 is small, decoding is afordably fast, and for the encoding scheme the security can be increased by increasing the n_w.

But n_w can not be too big, because Groebner attacks will break the system. Code rates 1:4, 1:5, 1:6

Security analysis

3. Rank Attacks (on the code and on the dual code)



For encryption, k_1 values in range 20 – 34 and code rates 1:4, 1:5, 1:6 offer security against known attacks in range 2^{80} - 2^{160}

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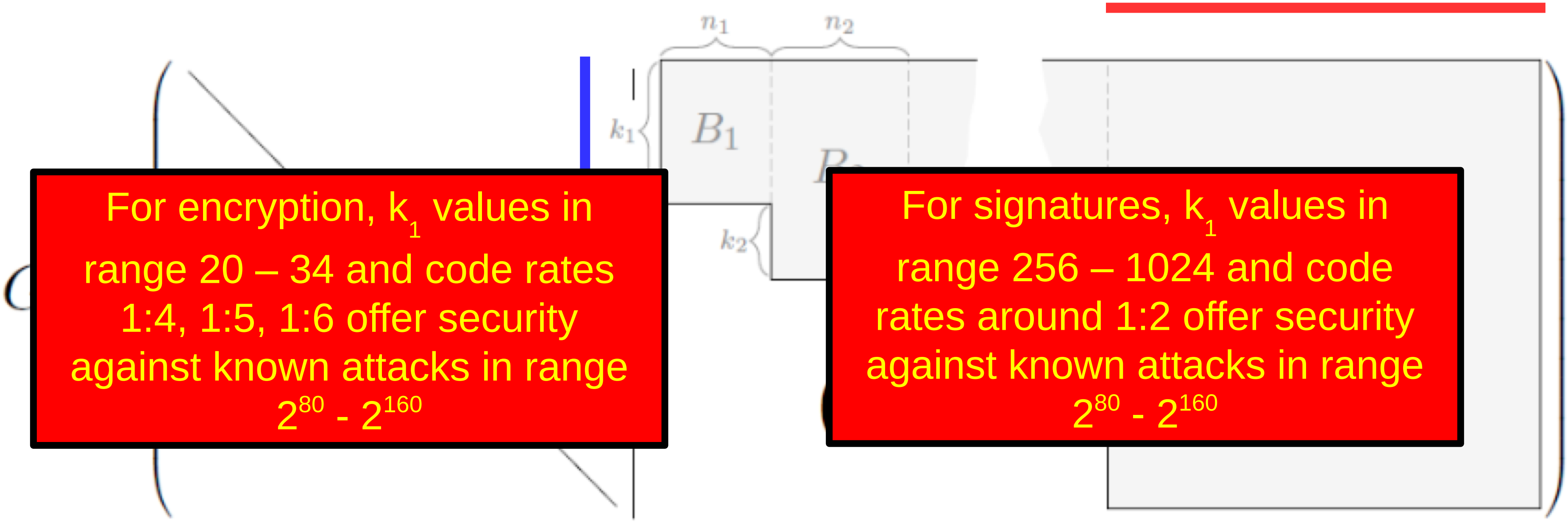
3. Rank Attacks (on the code and on the dual code)



For encryption, k_1 values in range 20 – 34 and code rates 1:4, 1:5, 1:6 offer security against known attacks in range 2^{80} - 2^{160}

For signatures, k_1 values in range 256 – 1024 and code rates around 1:2 offer security against known attacks in range 2^{80} - 2^{160}

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3. Rank Attacks (on the code and on the dual code)

$$Pr_{rank} = \binom{n/\ell - (K_t/\ell + 1)}{N_t/\ell - (K_t/\ell + 1)} \binom{n/\ell}{N_t/\ell}^{-1}$$

The rank computation takes approximately $k(K_t + \ell)^{\omega - 1}$ operations, where ω is the linear algebra constant.

Table 3. Concrete complexity of rank attack for $\ell = 2$, $\rho = 3^{1/2}$.

k	256	512	768	1024	1280	1536	1792	2048
K_1	20	22	24	26	28	30	32	34
Complexity of the attack	$2^{82.9}$	$2^{103.7}$	$2^{118.9}$	$2^{132.4}$	$2^{144.9}$	$2^{156.9}$	$2^{168.5}$	2^{180}

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4. Cheap distinguishers or equivalent keys finders (brought to us by Nicolas Sendrier)

From the elements in L_1 we build up the temporary list T_1 of all possible decodings of $\mathbf{y}_0$ having length $4 + 1 = 5$:

T_1

Repeating the above proc

Unlike other code-based systems, if you try to find an equivalent key (by some more efficient distinguishers than those discussed in part 3.), you have to pay attention your key to keep this ratios in order to be usefull for the list-decoding.

$L_3 = \{(101001)\}$.

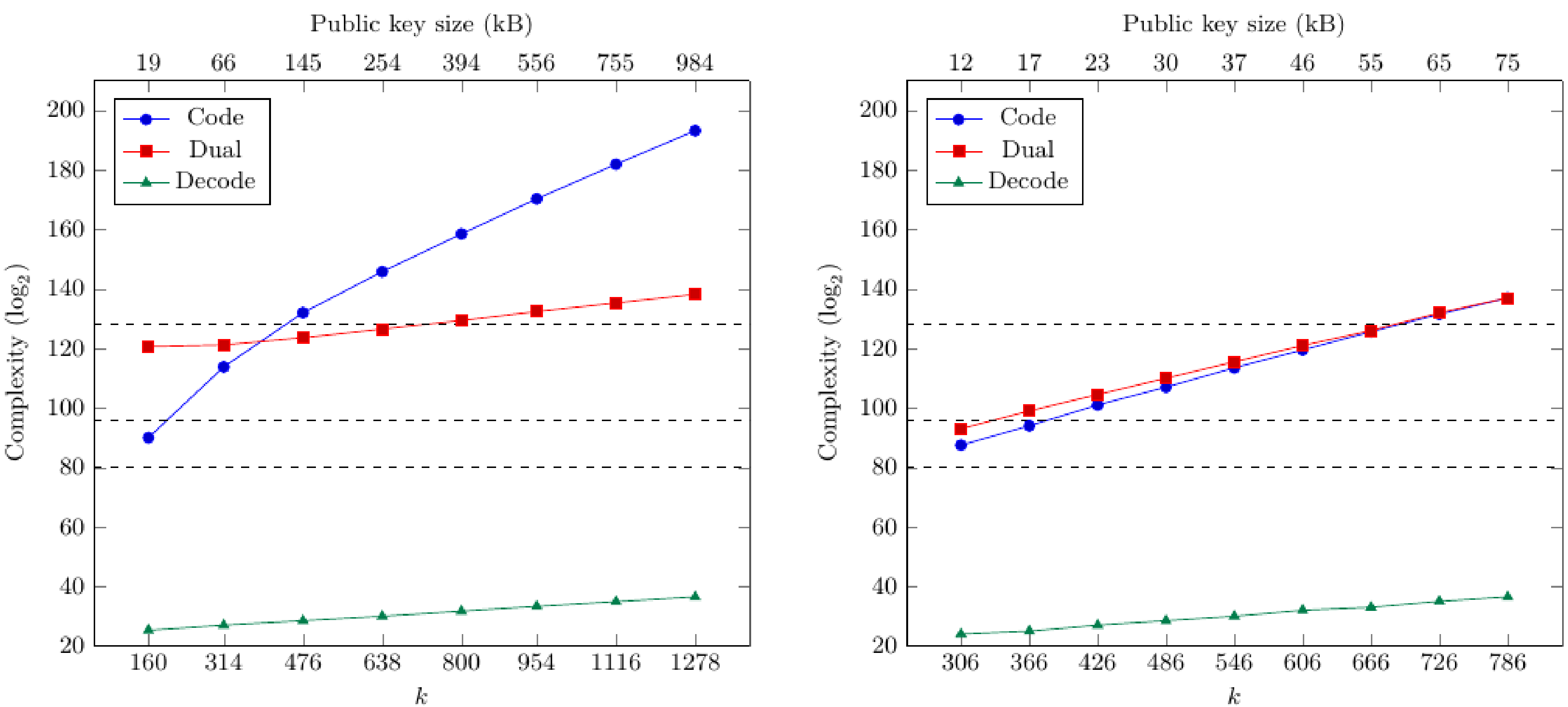
Thus, in this case we obtain a unique decoding.

The efficiency of the list decoding algorithm depends on the size of the lists $L_0, L_1, \ldots, L_w$, and whether during the decoding process each new list has a smaller size than the previous one. If the size of the lists decreases, the overall complexity is dominated by the size of the initial list L_0. Therefore, given a parameter k_1 (which determines L_0), we want to impose constrains on the values of n_i/k_i in order to avoid "blow-up" of the list sizes.

Proposition 3. *Let* $\mathrm{E}[|L_i|]$ *denote the expected value of the size of the lists* $L_1, L_2, \ldots, L_w$. *Then* $|L_0| \geq \mathrm{E}[|L_1|] \geq \cdots \geq \mathrm{E}[|L_w|]$ *if and only if* $\dfrac{n_i}{k_i} \geq \dfrac{\log_2 \rho}{1 - \log_2 \rho}$ *for all* $2 \leq i \leq w$.

6 Choosing Parameters

One important issue with any cryptographic primitive is its efficiency for a given level of claimed security. For public-key primitives, this can be examined by analyzing the sizes of the private and public key, and the number of operations necessary for encryption, decryption, signing and verification.



(a) Encryption (b) Signature

Fig. 5. Comparison between the complexity of decoding and the distinguishing attacks for encryption and signature. Dashed horizontal lines denote three security levels: 2^{80}, 2^{96} and 2^{128}.

Some concrete parameter proposals

Variant	Security level	n	k	Public key size [KB]	Decoding operations
Encryption	2^{80}	1280	256	40	2^{19}
Encryption	2^{96}	2560	512	160	$2^{20.6}$
Encryption	2^{128}	5120	1024	640	$2^{23.8}$
Signature	2^{80}	1208	256	37.7	2^{19}
Signature	2^{96}	2416	512	151	$2^{20.6}$
Signature	2^{128}	4832	1024	604	$2^{23.8}$

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Potentials of these Staircase-Generator Codes for other applications

- *Approaching Maximum Embedding Efficiency on Small Covers Using Staircase-Generator Codes*, S. Samardjiska and D. Gligoroski, 2015 IEEE International Symposium on Information Theory, June 14-19, 2015, Hong Kong
- By applying a simmilar approach as in the signature variant of the public key scheme, but used for **steganographic matrix embedding**, these codes achieve almost the upper theoretical bound of the embedding efficiency for sizes of the covers in the range of 1000 – 1500.
- Other steganographic schemes based on matrix embedding that offer embedding efficiency close to the theoretical bound are based on the low-density generator matrix (LDGM) codes, and they achieve that bound for large covers in the range $10^5 - 10^6$.
- These Staircase-Generator Codes achieve the upper theoretical bound with **two or three orders of magnitude** smaller covers.

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Thank you for your attention!

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