2F - A New Method for Constructing Efficient Multivariate Encryption Schemes

Daniel Smith-Tone[1,2]

[1]University of Louisville
[2]National Institute of Standards and Technology

16 November, 2022

Objective

Given a multivariate quadratic system of equations

$$P(\mathbf{x}) = \mathbf{y},$$

find $\mathbf{x}$.

Direct Attack

- Solve directly via F4 or XL.
  (Consider the Macaulay matrix: rows = equations, columns = monomials.)
- Complexity related to homogeneous quadratic component.
- Field Equations $(x_i^q - x_i)$
- With hybrid approach we consider the Hilbert series

$$\mathcal{H}(t) = \frac{(1-t^2)^m(1-t^q)^{n-k}}{(1-t)^{n-k}}$$

Differential Attacks

Idea that broke SFLASH. (Also breaks, C^*, k-ary C^*, ℓIC-, etc.)
Discrete Differential $DP(\mathbf{a},\mathbf{x}) = P(\mathbf{a}+\mathbf{x}) - P(\mathbf{a}) - P(\mathbf{x}) + P(\mathbf{0})$.

$$DP(L\mathbf{a},\mathbf{x}) + DP(\mathbf{a},L\mathbf{x}) = \Lambda_L DP(\mathbf{a},\mathbf{x})$$

Rank Attacks

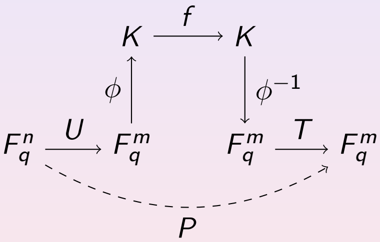
Minrank: Given K matrices $\mathbf{M}_1, \ldots, \mathbf{M}_K$ of dimension $s \times t$ over the field F, find nonzero coefficients $\lambda_1, \ldots, \lambda_k$ in the field E/F such that

$$\text{rank}\left(\sum_{i=1}^{K} \lambda_i \mathbf{M}_i\right) \leq r.$$

Multivariate Encryption Schemes

Scheme	PK	pt	ct	Enc.(ms)	Dec.(ms)
ABC(2^8,384,760)	54863KB	384B	760B	502	545
PCBM(149,414)	743KB	149b	414b	13	743

Definition of SQUARE



U is injective, $f(X) = X^2$, q odd prime-power.

Attacks

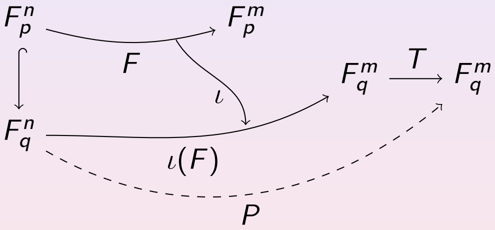
- Direct Attack
- Differential Attack (Perturb Input recover in output)
- Differential Attack (Perturb Output recover in input)
- Rank Attack (Big field "traditional")
- Rank Attack (Big field, Tao et al. style)

Linear Maps are Important

Something critical in all of these attacks (or their analyses) is the role of linear maps.

Question: Can we augment a quadratic map in a nonlinear way to disrupt these cryptanalyses?

Modulus Switching



Example

Let $p = 7$ $n = m = 3$ and $q = 331$.

$$\begin{aligned} v_1 &= 2x_1^2 - x_1x_2 - 2x_1x_3 + 0x_2^2 + 3x_2x_3 - x_3^2 \\ v_2 &= x_1^2 + 3x_1x_2 - x_1x_3 - 3x_2^2 + 0x_2x_3 - 2x_3^2 \\ v_3 &= -x_1^2 - 3x_1x_2 + x_1x_3 + 2x_2^2 - x_2x_3 + x_3^2 \end{aligned}$$

Example

Let $p = 7$ $n = m = 3$ and $q = 331$.

$$
\begin{aligned}
v_1 &= 2(1)^2 - (1)(-2) - 2(1)(2) + 0(-2)^2 + 3(-2)(2) - (2)^2 \\
v_2 &= (1)^2 + 3(1)(-2) - (1)(2) - 3(-2)^2 + 0(-2)(2) - 2(2)^2 \\
v_3 &= -(1)^2 - 3(1)(-2) + (1)(2) + 2(-2)^2 - (-2)(2) + (2)^2
\end{aligned}
$$

Example

Let $p = 7$ $n = m = 3$ and $q = 331$.

$$v_1 = -2$$
$$v_2 = 1$$
$$v_3 = 2$$

Example

Let $p = 7$ $n = m = 3$ and $q = 331$.

$$v_1 = -16$$
$$v_2 = -27$$
$$v_3 = 23$$

Example

Let $p = 7$ $n = m = 3$ and $q = 331$.

$$v_1 = -16$$
$$v_2 = -27$$
$$v_3 = 23$$

$$y_1 = -153$$
$$y_2 = -83$$
$$y_3 = 109$$

Why it Works

If

$$q > \frac{(p-1)^3}{4}\binom{n+1}{2},$$

then $\mathbf{y} = T \circ \iota(F)(\mathbf{x})$ if and only if $T^{-1}(\mathbf{y}) = F(\mathbf{x}) \pmod{p}$.

Decryption Failures

$$q > \frac{(p-1)^3}{4}\binom{n+1}{2} \Rightarrow \text{no } \textit{new} \text{ decryption failures.}$$

These quadratic distributions are rather tight, so much smaller q are possible.
If we further restrict $x_i \in \{-1, 0, 1\}$, the distributions are even tighter. Can have much larger $p < q$.

Direct Attack

Instead of field equations, we have

$$g_i(x_i) = \prod_{j=\frac{1-p}{2}}^{\frac{p-1}{2}} (x_i - j).$$

$$\mathcal{H}(t) = \frac{(1-t^2)^m(1-t^p)^{n-k}}{(1-t)^{n-k}}$$

If $x_i \in \{-1, 0, 1\}$, then

$$\mathcal{H}(t) = \frac{(1-t^2)^m(1-t^3)^{n-k}}{(1-t)^{n-k}}$$

Differential Attacks

$$DP(\mathbf{La},\mathbf{x}) + DP(\mathbf{a},\mathbf{Lx}) = \mathbf{\Lambda_L} DP(\mathbf{a},\mathbf{x})$$

F_p-linear

Need F_p-linear

Also need F_q-linear

Rank Attacks

For small field schemes, rank structure may be preserved.
For big field schemes,

$$[\mathbf{H}_1\ \mathbf{H}_2\ \cdots\ \mathbf{H}_m](\mathbf{M}\otimes\mathbf{I}_m)=\left[\mathbf{S}\mathbf{G}^{*0}\mathbf{S}^\top\ \cdots\ \mathbf{S}\mathbf{G}^{*(n-1)}\mathbf{S}^\top\right],$$

where $\mathbf{H}_i$ is the ith quadratic form of the hidden quadratic map.
The problem is

$$[P_1\ P_2\ \cdots\ P_m]=\left[\widetilde{\mathbf{H}}_1\ \widetilde{\mathbf{H}}_2\ \cdots\ \widetilde{\mathbf{H}}_m\right](\mathbf{T}\otimes\mathbf{I}_m).$$

Lattice Attacks

Let $\mathbf{P}$ be the Macaulay matrix of the public key P.
$\mathbf{P}$ is $m \times \binom{n+1}{2}$.
Consider

$$\begin{bmatrix} \frac{p}{q}\mathbf{I}_m & \mathbf{P} \\ \mathbf{0} & q\mathbf{I}_{\binom{n+1}{2}} \end{bmatrix}.$$

Ray Perlner has a much better lattice-based attack. (Breaks parameters from paper.)

Recall that we can restrict $x_i \in \{-1, 0, 1\}$ and use much larger p and smaller q.

Getting the Right Dimension

The Macaulay matrix $\mathbf{P}$ defines a lattice with a very large dimension, but a small rank.

$$\text{Let } \mathbf{P} = \begin{bmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \end{bmatrix},$$

where $\mathbf{B}, \mathbf{C}$ are $m \times m$.

$$\begin{bmatrix} \mathbf{I}_m & \mathbf{A}^{-1}\mathbf{B} \\ \mathbf{0} & q\mathbf{I}_m \end{bmatrix}$$

Making Long Vectors

By the Gaussian Heuristic, we expect the shortest vector to be of length

$$gh(\mathcal{L}) = \sqrt{d/2\pi e}\mathrm{Vol}(\mathcal{L})^{1/d}.$$

With $d = 2m$ and $\mathrm{Vol}(\mathcal{L}) = q^m$, we get

$$\text{Length of shortest vector} \approx \sqrt{mq/\pi e}.$$

Expected length of Macaulay vector is

$$\sqrt{8m/(p-1)} \sum_{i=0}^{(p-1)/2} i^2.$$

Use SQUARE

Most "standard" multivariate attacks can be used to break SQUARE.

Goal: Create weakest possible target to test the 2F construction.

Parameters and Perfomance in Article

Scheme	PK	pt	ct	Enc.(ms)	Dec.(ms)
ABC(2^8,384,760)	54863KB	384B	760B	502	545
PCBM(149,414)	743KB	149b	414b	13	743
2FSQ(3, 6653, 81)	417KB	162b	129B	1.5	0.4
2FSQ(3, 8377, 91)	606KB	182b	148B	1.2	0.5
2FSQ(7, 130411, 69)	346KB	207b	147B	1.0	2.6
2FSQ(7, 145861, 73)	413KB	219b	157B	1.1	2.8

BROKEN

Parameters and Perfomance

Scheme	PK	pt	ct	Enc.(ms)	Dec.(ms)
ABC(2^8,384,760)	54863KB	384B	760B	502	545
PCBM(149,414)	743KB	149b	414b	13	743
2FSQ(67, 6247, 93)	626KB	186b	147B	1.3	16.75

Profile

- Small ciphertexts
- Large public keys
- Fairly slow decryption

Future Directions

1) More security analysis.

2) Examine 2F applied to other schemes.