A Subexponential Quantum Algorithm for the Semidirect Discrete Logarithm Problem

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Motivation

Group-based Cryptography

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Non-interactive Key Exchange (NIKE)

- Use cases for resource-constrained settings vs KEMs
- Not many such schemes available

A Novel Exponentiation

Definition

- Let G a finite, non-abelian (semi)group and $End(G)$ its induced semigroup of endomorphisms
- Each $(g, \phi) \in G \times End(G)$ induces a function $s_{g,\phi} : \mathbb{N} \to G$

$$s_{g,\phi}(x) = \phi^{x-1}(g) \cdot \ldots \cdot \phi(g) \cdot g$$

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$$\phi^{x-1}(g) \cdot \ldots \cdot g \longrightarrow \boxed{\phi^{y}} \longrightarrow \phi^{x+y-1}(g) \cdot \ldots \cdot \phi^{y}(g)$$

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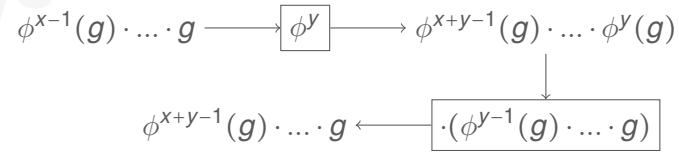

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There is a function $* : \mathbb{N} \times \mathcal{X}_{g,\phi} \to \mathcal{X}_{g,\phi}$, pronounced *step*, such that

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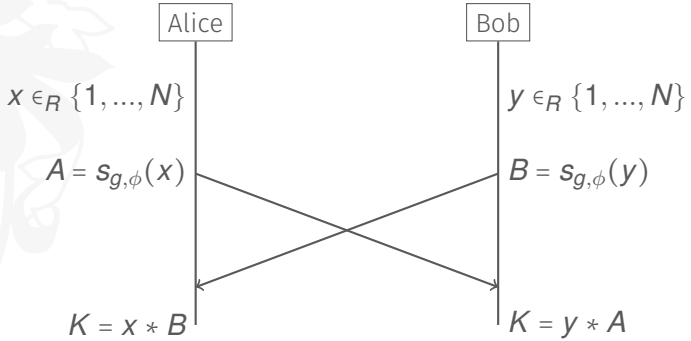
Indeed

$$y * s_{g,\phi}(x) = s_{g,\phi}(x + y) = x * s_{g,\phi}(y)$$

Semidirect Product Key Exchange (SDPKE)

Public parameters:
$G, (g, \phi), N = |\mathcal{X}_{g,\phi}|$



Habeeb, Kahrobaei, Koupparis, and Shpilrain. "Public key exchange using semidirect product of (semi) groups". In: *ACNS*. Springer

Non-interactive SDPKE)

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Alice	Bob
$sk_A = x \in_R \{1, ..., N\}$	$sk_B = y \in_R \{1, ..., N\}$
$pk_A = s_{g,\phi}(x)$	$pk_B = s_{g,\phi}(y)$
$K = x * pk_B$	$K = y * pk_A$

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Definition (SDLP)

Let $(g, \phi) \in G \times End(G)$ and x sampled uniformly at random from $\{1, ...N\}$. One solves the *Semidirect Discrete Logarithm Problem* (SDLP) if one can recover x from $(g, \phi), s_{g,\phi}(x)$.

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Definition (SCDH)

Let $(g, \phi) \in G \times End(G)$ and x, y sampled uniformly at random from $\{1, ..., N\}$. One solves *Semidirect Computational Diffie-Hellman* (SCDH) if one can recover $s_{g,\phi}(x + y)$ from $(g, \phi), s_{g,\phi}(x), s_{g,\phi}(y)$.

The Vectorisation Problem

Definition (Group Actions)

A *group action* is a triple $(G, X, \star)$ where G is a group, X is a set, and $\star : G \times X \to X$ is a function such that:

- $1 \star x = x$ for all $x \in X$
- $(a + b) \star x = a \star (b \star x)$ for all $a, b \in G$, $x \in X$

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Definition (Vectorisation Problem)

Let $(G, X, \star)$ a finite group action. Given $x, y \in X$, the *Vectorisation Problem* is to find $g \in G$ such that $g \star x = y$.

Jean-Marc Couveignes. "Hard homogeneous spaces". In: *Cryptology ePrint Archive* (2006)

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- We construct a quantum algorithm that solves SDLP in subexponential time
- We show the following three computational problems are quantum equivalent: SDLP, SCDH, and Couveignes' *Vectorisation Problem*

Tail and Cycle

Let $\mathcal{X}_{g,\phi} = \{s_{g,\phi}(i) : i \in \mathbb{N}\}$. There is a function $* : \mathbb{N} \times \mathcal{X}_{g,\phi}$ such that

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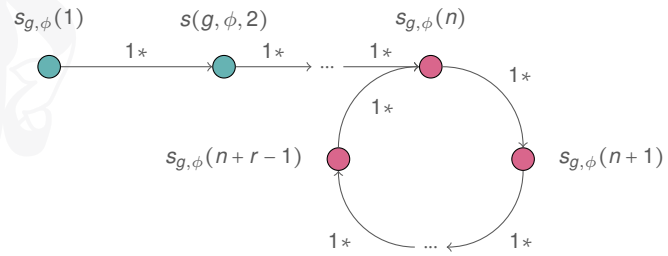

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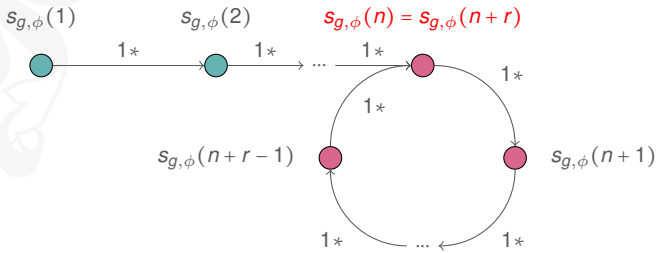

$$s_{g,\phi}(n + r + 1) = 1 * s_{g,\phi}(n + r) = 1 * s_{g,\phi}(n) = s_{g,\phi}(n + 1)$$

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Theorem

A cyclic group of size $r_{g,\phi}$, denoted $\mathbb{N}_r$, acts on the cycle $\mathcal{C}_{g,\phi}$.

Solving SDLP

Theorem

One can solve SDLP in quantum subexponential time.

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- $s_{g,\phi}(x)$ is in the cycle if and only if $r_{g,\phi} \star s_{g,\phi}(x) = s_{g,\phi}(x)$

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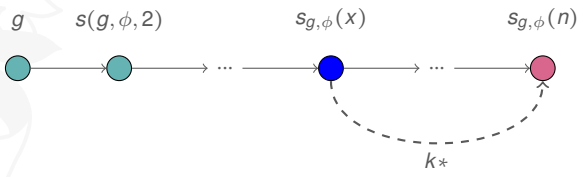

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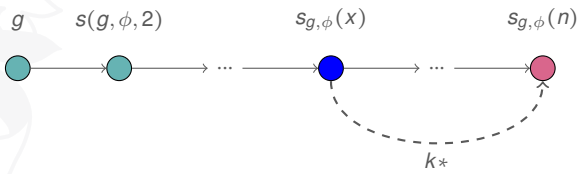

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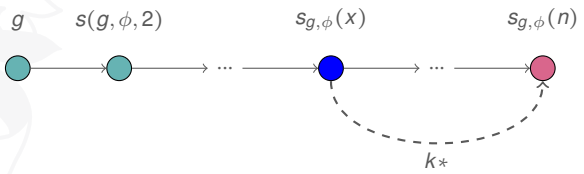

- Find the smallest j such that $j * s_{g,\phi}(x)$ is invariant under $r*$ - can be done efficiently with binary search
- This value, say k, is such that $n - k = x$.

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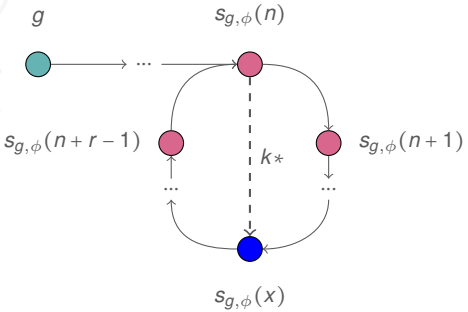

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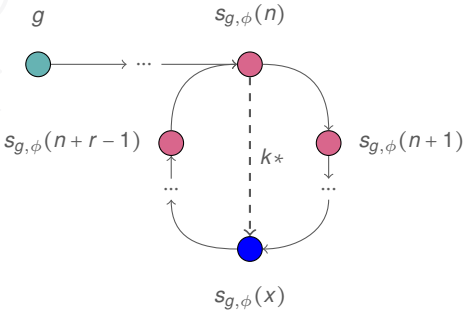

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- This group element tells us k such that $k * s_{g,\phi}(n) = s_{g,\phi}(x)$, so recover $x = n + k$.

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$$\sum_{j=0}^{M-1} |j\rangle |s_{g,\phi}(j)\rangle$$

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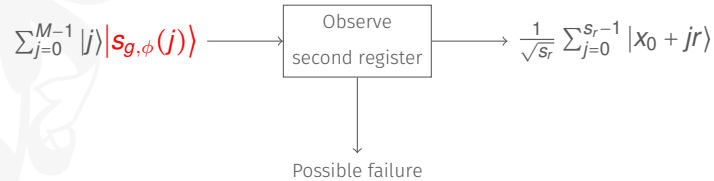

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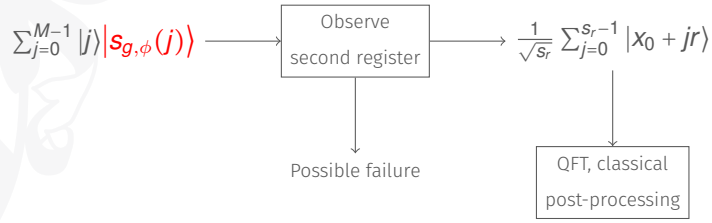

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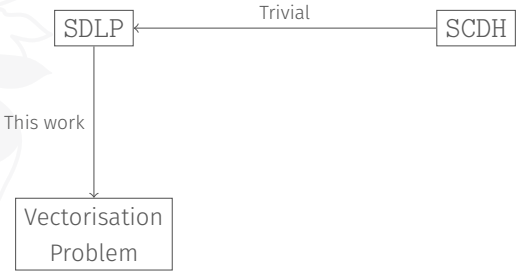
The Landscape of Problems



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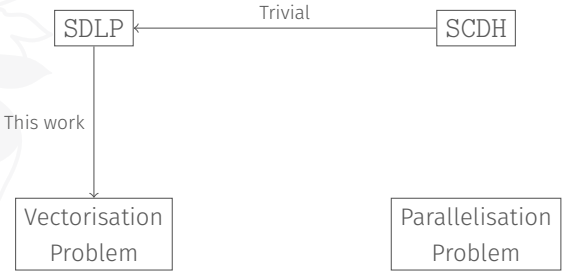

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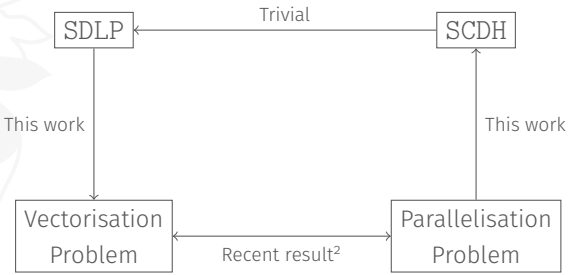

[2] Hart Montgomery and Mark Zhandry. "Full quantum equivalence of group action DLog and CDH, and more". In: *ASIACRYPT* (2022).

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- Our framework is a rare non-trivial example of a cryptographically useful group action
- Lots of further study required!

The End.

Thank you.