QC-MDPC-McEliece: A public-key code-based encryption scheme based on quasi-cyclic moderate density parity check codes

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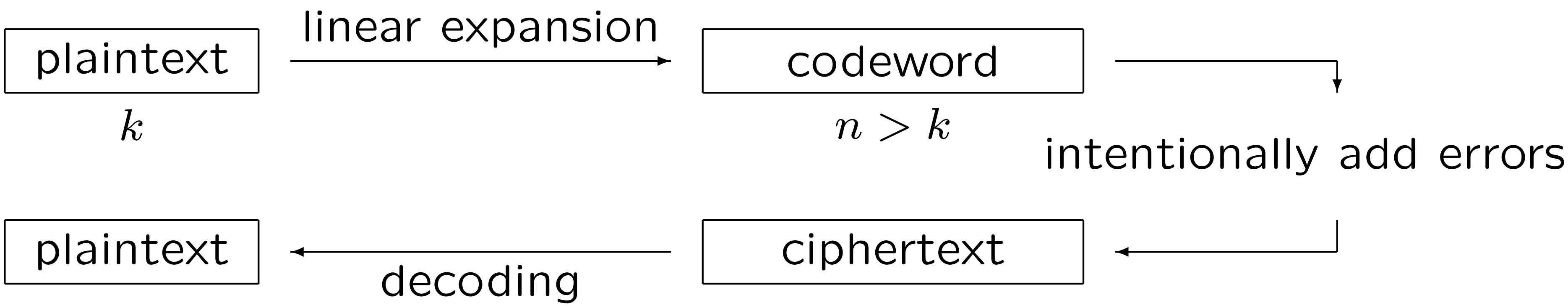
Error Correcting Codes for Public-Key Encryption



- If a random linear expansion is used, no one can decode efficiently
- If a "good" error correcting code is used for the expansion, anyone who knows the structure has access to a fast decoder

Assuming that the knowledge of the linear expansion does not reveal the code structure:

- The linear expansion is public and anyone can encrypt
- The decoder is known to the legitimate user who can decrypt
- For anyone else, the public linear expansion looks random

McEliece Public-key Encryption Scheme – Overview

$\mathcal{F}$ a family of t-error correcting binary linear $[n, k]$ code

Key generation:

$$\mathcal{C} \in \mathcal{F} \to \begin{cases} \textbf{Public Key: } G \in \{0,1\}^{k\times n}, \text{ a generator matrix} \\ \textbf{Secret Key: } \Phi : \{0,1\}^n \to \mathcal{C}, \text{ a decoder correcting } t \text{ errors} \end{cases}$$

Encryption: $\left[\begin{array}{rccc} E_G : & \{0,1\}^k & \to & \{0,1\}^n \\ & x & \mapsto & xG + e \end{array}\right]$ with e random of weight t

Decryption: $\left[\begin{array}{rccc} D_\Phi : & \{0,1\}^n & \to & \{0,1\}^k \\ & y & \mapsto & \Phi(y)G^* \end{array}\right]$ where $GG^* = 1$

[McEliece, 1978] $\mathcal{F}$ is a family of binary Goppa codes

$n = 1024$, $k = 524$, $t = 50$

Hardness of Decoding

[Berlekamp, McEliece, & van Tilborg, 78]

Syndrome Decoding	NP-complete
Instance: $H \in \{0,1\}^{(n-k)\times n}$, $s \in \{0,1\}^{n-k}$, w integer	
Question: Is there $e \in \{0,1\}^n$ such that $\mathrm{wt}(e) \leq w$ and $eH^T = s$?	

[Alekhnovich, 03]

Conjectured difficult on average for $w = n^\varepsilon$ and any $\varepsilon > 0$

Best known decoder for w errors in an $[n,k]$ code has complexity

$$W_{\mathrm{SD}}(n,k,w) = 2^{(c+o(1))w \log_2 \frac{n}{n-k}}$$

[Prange, 62] Information Set Decoding, $c \approx 1.1$ when $w = \theta(n)$
[Becker & Joux & May & Meurer, 12] $c \approx 0.9$ when $w = \theta(n)$

When $w = o(n)$ then $c = 1$ for all classical variants of ISD

[Bernstein, 09] quantum computing $\rightarrow$ c divided by 2 (at most)

Security Reduction

For given parameters n, k, and t
$\mathcal{K} = \{0, 1\}^{k \times n}$ the "apparent" key space
$\mathcal{G} \subset \mathcal{K}$ the set of all public keys

Theorem

If there exists an efficient *adversary* against McEliece then

- either there exists an efficient *distinguisher* for $\mathcal{G}$ *versus* $\mathcal{K}$
- or there exists an efficient *generic decoder* for t errors in $[n, k]$ codes

In other words, if we assume that

1. $\mathcal{G}$ is pseudorandom
2. decoding is hard on average

then McEliece's scheme (with public keys in $\mathcal{G}$) is secure "on average"

$+$ a semantically secure conversion $\rightarrow$ any desirable security level

More on Semantic Security

Because the scheme is malleable (replay attack [Berson, 97], reaction attack [Kobara & Imai, 00]) a semantically secure conversion is mandatory

First semantically secure conversion: [Kobara & Imai, 01]

With a semantic security layer the public key can be in systematic form [Biswas & S.,08]

$$G = \begin{array}{|ccc|c|}\hline 1 & & & \\ & \ddots & & \\ & & 1 & \\ \hline \end{array}$$

$\rightarrow$ smaller key size, easier encryption

Quasi-Cyclic instances of McEliece's Scheme (1/2)

(similar to NTRU, Ring LWE, ideal lattices)

The public key is formed of circulant blocks, for instance:

$$G = \left[\begin{array}{c|c} \begin{matrix}1 & \\ & 1\end{matrix} & \boxed{\;\;g\;\;} \end{array}\right]$$

$$G = \left[\begin{array}{c|c|c|c|c} \begin{matrix}1 & \\ & 1\end{matrix} & & \boxed{g_{0,0}} & \boxed{g_{0,1}} & \boxed{g_{0,2}} \\ \hline & \begin{matrix}1 & \\ & 1\end{matrix} & \boxed{g_{1,0}} & \boxed{g_{1,1}} & \boxed{g_{1,2}} \end{array}\right]$$

Advantage: much smaller key size

Difficulty: hide the code structure (*i.e.* the secret decoder)

Quasi-Cyclic instances of McEliece's Scheme (2/2)

- Goppa (or alternant) codes, initiated by [Gaborit, 05]

  Too much algebraic structure, some attempts have failed, to be used with care

- "Disguised" LDPC (Low Density Parity Check) codes [Baldi & Chiaraluce, 07]

  Less structure but still no convincing security reduction

- MDPC (Moderate Density Parity Check) codes [Misoczki & Tillich & S. & Barreto, 13]

  Even less structure, a security reduction

[Misoczki & Barreto, 09]
Also possible with dyadic blocks instead of circulant blocks

MDPC McEliece

QC-MDPC-McEliece Scheme (1/2)

Parameters: n, k, w, t
(for instance $n = 9601$, $k = 4801$, $w = 90$, $t = 84$)

Key generation: (rate $1/2$, $n = 2p$, $k = p$)

Pick a (sparse) vector $(h_0, h_1) \in \{0,1\}^p \times \{0,1\}^p$ of weight w

$$H_{\text{secret}} = \left[\begin{array}{c|c} \boxed{\quad h_0 \quad} & \boxed{\quad h_1 \quad} \\ \circlearrowright & \circlearrowright \end{array}\right]$$

with $h_0(x)$ invertible in $\mathbf{F}_2[x]/(x^p - 1)$
(circulant binary $p \times p$ matrices are isomorphic to $\mathbf{F}_2[x]/(x^p - 1)$)

Publish $h(x) = h_1(x) h_0^{-1}(x) \mod x^p - 1$ or $g(x) = \overline{h(x)/x}$

$$H = \left[\begin{array}{cc|c} 1 & & \boxed{\quad h \quad} \\ & 1 & \circlearrowright \end{array}\right] \text{ or } G = \left[\begin{array}{c|cc} \boxed{\quad g \quad} & 1 & \\ \circlearrowright & & 1 \end{array}\right]$$

H a parity check matrix, G a generator matrix

QC-MDPC-McEliece Scheme (2/2)

Encryption: (rate 1/2, $n = 2p$, $k = p$)

$$\begin{array}{rcl} \mathbf{F}_2[x]/(x^p-1) & \rightarrow & \mathbf{F}_2[x]/(x^p-1) \times \mathbf{F}_2[x]/(x^p-1) \\ m(x) & \mapsto & (m(x)g(x) + e_0(x), m(x) + e_1(x)) \end{array}$$

The error $e(x) = (e_0(x), e_1(x))$ has weight t

Decryption:

Iterative decoding (as for LDPC codes) which only requires the sparse parity check matrix. For instance the "bit flipping" algorithm

Parameters are chosen such that the decoder fails to correct t errors with negligible probability

Each iteration has a cost proportional to $w \cdot (n-k)$, the number of iterations is small (3 to 5 in practice)

QC-MDPC-McEliece Security Reduction

$$H = \left[\begin{array}{cc|c} 1 & & \boxed{\quad h \quad} \\ & \ddots & \\ & 1 & \circlearrowright \end{array}\right] \text{ with } h(x) = \frac{h_1(x)}{h_0(x)} \mod x^p - 1$$

Secure under two assumptions

1. Pseudorandomness of the public key

   Hard to decide whether there exists a sparse vector in the code spanned by H (the dual of the MDPC code)

2. Hardness of generic decoding of QC codes

   Hard to decode in the code of parity check matrix H (for an arbitrary value of h)

Sparse Polynomial Problems

The security reduction and the attacks can be stated in terms of polynomials

1. **Key Security**

   Given $h(x)$, find non-zero $(h_0(x), h_1(x))$ such that

   $$\begin{cases} h_0(x) + h(x)h_1(x) = 0 \mod x^p - 1 \\ \mathrm{wt}(h_0) + \mathrm{wt}(h_1) \leq w \end{cases}$$

   or simply decide the existence of a solution $\rightarrow$ distinguisher

2. **Message Security**

   Given $h(x)$ and $S(x)$, find $e_0(x)$ and $e_1(x)$ such that

   $$\begin{cases} e_0(x) + h(x)e_1(x) = S(x) \mod x^p - 1 \\ \mathrm{wt}(e_0) + \mathrm{wt}(e_1) \leq t \end{cases}$$

In both cases, best known solutions use generic decoding algorithms

Practical Security – Best Known Attacks

Let $W_{\mathsf{SD}}(n, k, t)$ denote the cost for the generic decoding of t errors in a binary $[n, k]$ code

We consider a QC-MDPC-McEliece instance with parameters n, k, w, t and circulant blocks of size p.

1. **Key Attack:** find a word of weight w in a quasi-cyclic binary $[n, n-k]$ code

$$W_K(n, k, w) \geq \frac{W_{\mathsf{SD}}(n, n-k, w)}{n-k}$$

(there are $n-k$ words of weight w)

2. **Message Attack:** decode t errors in a quasi-cyclic binary $[n, k]$ code

$$W_M(n, k, t, p) \geq \frac{W_{\mathsf{SD}}(n, k, t)}{\sqrt{p}}$$

(Decoding One Out of Many [S., 11] $\rightarrow$ factor $\sqrt{p}$)

Parameter Selection

Choose a code rate k/n and a security exponent S (for instance 80 or 128). Then increase the block size until the following succeeds:

- find w the smallest integer such that $W_K(n, k, w) \geq 2^S$
- find t the error correcting capability of the corresponding MDPC code
- check that $W_M(n, k, t, p) \geq 2^S$

80 bits of security	128 bits of security
$n = 9602$	$n = 19714$
$k = 4801$	$k = 9857$
$p = 4801$	$p = 9857$
$w = 90$	$w = 142$
$t = 84$	$t = 134$

Scalability

A binary $[n, k]$ code with $n-k$ parity equations of weight w will correct t errors with an LDPC-like decoding algorithm as long as $t \cdot w \lessapprox n$

For LDPC codes, we have essentially $w = O(1)$. For MDPC codes we have $w = O(\sqrt{n})$ and thus $t = O(\sqrt{n})$.

The optimal trade-off between the key size (K) and the security (S) is obtained for codes of rate $1/2$ and

$$K \approx cS^2 \text{ with } c < 1$$

For Goppa code, the optimal code rate is ≈ 0.8 and

$$K \approx c\,(S \log_2 S)^2 \text{ with } c \approx 2$$

Conclusion

QC-MDPC-McEliece is a promising variant which enjoys

- a reasonable key size
- good security arguments (very little structure)
- secure against quantum computers
- easy implementation (including lightweight implementation) [Heyse & von Maurich & Güneysu, 13]

Thank you for your attention

Bit-Flipping Decoding

Parameter: a threshold T

> input: $y \in \{0,1\}^n$, $H \in \{0,1\}^{(n-k)\times n}$
>
> Repeat
>
> Compute the syndrome Hy^T
>
> for $j = 1, \dots, n$
>
> if more than T parity equations involving j are violated then
>
> flip y_j

$Hy^T = \begin{pmatrix} s_1 \\ \vdots \\ s_{n-k} \end{pmatrix}$, if $s_i \neq 0$ the i-th parity equation is violated

If H is sparse enough and y close to the code of parity check matrix H then the algorithm finds the closest codeword after a few iterations